UNIVERSITY OF ALBERTA

Results in Anticipative Stochastic Calculus

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A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLEMT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY

EDMONTON, ALBERTA
Fall 1992

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ISBN 0-315-77412-6

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TITLE OF THESIS: **Results in Anticipative Stochastic Calculus**

DEGREE: Doctor of Philosophy

YEAR THIS DEGREE GRANTED: Fall 1992

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ABSTRACT

The theory of the stochastic anticipating calculus has been developed for several years. The purpose of this thesis is to derive further results in the subject. In Chapter 1 we review the basic properties of the Skorohod integral and the Malliavin derivative with respect to the Wiener process over some general parameter space T.

In Chapter 2 we consider a two-parameter stochastic process defined by the sum of multiple Skorohod integrals, a mixed Skorohod integral and ordinary Lebesgue integrals:

$$X_z = X_0 + \int_{R_z} f_\zeta d\zeta + \int_{R_z} \phi_\zeta dW_\zeta + \int_{R_z \times R_z} \psi(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} h(\zeta, \zeta') d\zeta dW_{\zeta'} + \int_{R_z \times R_z} g(\zeta, \zeta') d\zeta d\zeta'.$$

We establish a generalized Itô formula, using the Malliavin calculus techniques and the fundamental theorem of the differential calculus, essentially based on Ustunel's method for the one-parameter case.

The third chapter introduces the two-parameter Skorohod-Stratonovich integral and presents the Itô formula in the Stratonovich form.

The fourth chapter establishes a generalized Itô-Ventzell formula for two-parameter Skorohod integral processes when

$$X_z = X_0 + \int_{R_z} A_\zeta d\zeta + \int_{R_z} B_\zeta dW_\zeta$$

$$F_z(x) = F_0(x) + \int_{R_z} G_\zeta(x) d\zeta + \int_{R_z} H_\zeta(x) dW_\zeta.$$

In Chapter 5 an integral representation is derived for a class of stochastic processes with two-parameters. Consider a nonadapted square integrable process M satisfying:

$$E(M(z',z''] \mid \mathcal{F}_{(z',z'']^c}) = 0 \qquad \text{for any } z' < z'',$$

with $M_{s0} = 0$, $M_{0t} = 0$, where $M(z', z'']$ is the increment of M on $(z', z'']$. We obtain a necessary and sufficient condition for M to be expressed as an indefinite Skorohod integral, which extends the result of Duc and Nualart (1988) to the two-parameter case.

Using the method of the Wiener-Itô chaos decomposition, in Chapter 6, we construct a solution of the following linear Skorohod stochastic differential equation with two-parameters:

$$X_z = X_0 + \int_{R_z} A_\zeta X_\zeta d\zeta + \int_{R_z} B_\zeta X_\zeta dW_\zeta,$$

where X_0 is a random variable represented by a finite chaos decomposition, and A and B are square integrable deterministic functions.

In Chapter 7 we study a generalized multiple stochastic integral for non-adapted integrands following Hida's approach. We prove that the generalized stochastic integral $\int_{T^k} \partial^*_{s_k} \cdots \partial^*_{s_1} \varphi(\underline{s}) d\underline{s}$ coincides with the Skorohod integral $\delta^k \phi$ on $L^2(\Omega)$, where ∂^* denotes the adjoint of Hida operator ∂. For the two-parameter case, we give an Itô formula in white noise language. Furthermore we extend the white noise integral by Kuo and Russek to the two-parameter case and establish a series expansion of the Skorohod-Stratonovich integral.

Finally, in Chapter 8, we review the definition of the Skorohod integral over the Poisson space. By means of the Skorohod integral we provide a new proof of the representation formula for Poisson functionals.

ACKNOWLEDGEMENT

I sincerely thank my supervisor, Professor A. N. Al-Hussaini, for suggesting the topic for this thesis, and for spending many hours with me on its preparation. This work would have been impossible without his patience, constant support and advice. His encouragement and care also gave light to my life.

I also thank Drs. R.J. Elliott, K.L. Mehra, V. Perez-Abreu, B. Schmuland and V. Zizler for their helpful suggestions as committee members.

I would also like to thank Mrs. Carlean Fisher, Ms. Christine Fischer and Mrs. Laura Heiland for their help during my years at the University of Alberta and for teaching me how to type my thesis.

I am grateful to my parents and my husband for their continuous support during the years of my graduate education.

Last, but certainly not least, I gratefully acknowledge the Department of Statistics and Applied Probability for their continuing support during my studies, the University of Alberta for a Dissertation Fellowship and the Faculty of Graduate Studies and Research for a travel allowance to the 21st Conference on Stochastic Processes and Their Applications in June 1992.

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Chapter 1

PRELIMINARIES

In this chapter we recall the definitions of the Malliavin derivative and the multiple Skorohod integral, as well as their properties which we need for subsequent chapters. The main references are [2]-[5].

1.1. Definition of the derivative operator D^k

Let $(T, \mathcal{B}, \mu)$ be an atomless separable finite measure space. Consider a zero mean Gaussian process $\{W(B), B \in \mathcal{B}\}$ with covariance function given by $E(W(B_1)W(B_2)) = \mu(B_1 \cap B_2)$ defined in some probability space $(\Omega, \mathcal{F}, P)$. We assume that the σ-field $\mathcal{F}$ is generated by the random variables $W(B), B \in \mathcal{B}$. For any functional $F \in L^2(\Omega, \mathcal{F}, P)$, we have the Wiener-Itô chaos expansion [1]

$$F = \sum_{m=0}^{\infty} I_m(f_m), \qquad (1.1.1)$$

where $I_0(f_0) = EF$, and $I_m(f_m)$ denotes the multiple Itô integral of the non-random m-parameter and symmetric function $f_m \in L^2(T^m)$.

DEFINITION 1.1.1. Let $F \in L^2(\Omega, \mathcal{F}, P)$. For $k \geq 1$, the iterated Malliavin derivative D^kF is an element of $L^2(T^k \times \Omega)$ defined by

$$(D^kF)_s = \sum_{m=k}^{\infty} \frac{m!}{(m-k)!} I_{m-k}(f_m(s,\cdot)),$$

provided that this sum converges in $L^2(T^k \times \Omega)$.

It follows from the properties of the multiple Itô integral that

$$\|D^kF\|^2_{L^2(T^k \times \Omega)} = \sum_{m=k}^{\infty} [\frac{m!}{(m-k)!}]^2 \int_{T^k} E[I_{m-k}(f_m(s,\cdot))^2]\mu^k(ds)$$

$$= \sum_{m=k}^{\infty} \frac{(m!)^2}{(m-k)!} \|f_m\|^2_{L^2(T^m)}$$

where μ^k is the usual product measure on T^k. Therefore, the domain of D^k is the set

$$\text{Dom } D^k = \{ F \in L^2(\Omega): \sum_{m=k}^{\infty} \frac{(m!)2}{(m-k)!} \|f_m\|^2_{L^2(T^m)} < \infty \}$$

DEFINITION 1.1.2. For integers $n \geq 1$ and $k \geq 0$, let $\mathcal{H}^{(n)}(T^k \times \Omega) = \{u \in L^2(T^k \times \Omega)$: $u_s \in \text{Dom } D^j$, μ^k-a.s. s, and there exists a measurable version of $D^j_t u_s$ such that $D^j u \in L^2(T^{k+j} \times \Omega)$, $j = 1, 2, \dots, n\}$, where $\mathcal{H}^{(n)}(T^0 \times \Omega)$, simply denoted by $\mathcal{H}^{(n)}(\Omega)$, stands for Dom D^n.

Then $\mathcal{H}^{(n)}(T^k \times \Omega)$ is a Hilbert space with the inner product

$$(F, G)_{\mathcal{H}^{(n)}(T^k \times \Omega)} = (F, G)_{L^2(T^k \times \Omega)} + \sum_{j=1}^{n} (D^j F, D^j G)_{L^2(T^{k+j} \times \Omega)}$$

$$= \sum_{j=0}^{n} \sum_{p=j}^{\infty} \frac{(p!)^2}{(p-j)!} (f_p, g_p)_{L^2(T^{k+p})},$$

if $F_s = \sum_{p=0}^{\infty} I_p(f_p(s, \cdot))$, $G_s = \sum_{p=0}^{\infty} I_p(g_p(s, \cdot))$.

We also introduce some subspaces of $\mathcal{H}^{(n)}(T^k \times \Omega)$:

$\mathcal{H}^{p,n}(T^k \times \Omega) = \{ u \in \mathcal{H}^{(n)}(T^k \times \Omega)$: $D^j u \in L^p(T^{k+j} \times \Omega)$, $j = 0, 1, 2, \dots, n\}$,

for $n \geq 1$, $k \geq 0$ and $p \in [2, \infty)$, where $D^0 u$ stands for u.

PROPOSITION 1.1.3. Let $\Phi \in C^1(\mathbb{R}^r)$ be bounded, with bounded derivatives. Suppose $F^j \in \mathcal{H}^{(1)}(T^k \times \Omega)$, $j = 1, 2, \dots, r$. Then $\Phi(F^1, \cdots, F^r) \in \mathcal{H}^{(1)}(T^k \times \Omega)$ and

$$D_s \Phi (F^1_t, \cdots, F^r_t) = \sum_{j=1}^{r} \frac{\partial \Phi}{\partial x_j}(F^1_t, \cdots, F^r_t) D_s F^j_t .$$

PROPOSITION 1.1.4. *Let* $F, G \in Dom\ D$ *satisfying* $D^i F \cdot D^j G \in L^2(T^{i+j} \times \Omega)$ *for any* $i, j = 0, 1,\ i+j \leq 1$. *Then* $FG \in Dom\ D$, *and*

$$D_s(FG) = FD_sG + (D_sF)G.$$

Given $A \in \mathcal{B}$, we denote by $\mathcal{F}_A$ the σ- field generated by $W(B), B \in \mathcal{B}, B \subset A$.

PROPOSITION 1.1.5. *If* $F \in Dom\ D^k$, *then* $E(F \mid \mathcal{F}_A) \in Dom\ D^k$, *and*

$$D_t^k E(F \mid \mathcal{F}_A) = E(D_t^k F \mid \mathcal{F}_A) 1_{A^k}(t), \qquad a.e.$$

PROPOSITION 1.1.6. *Let* $F \in$ Dom D^k. *Then*

$$1_{\{F=0\}} D_t^k F = 0,\ d\mu^k(t) \times dP \ \ a.e.\ on\ T^k \times \Omega.$$

1.2. Definition of the Skorohod Integral

Let $u \in L^2(T^k \times \Omega)$ be a measurable process depending on k parameters. Fix $s \in T^k$, consider the representation

$$u_s = \sum_{m=0}^{\infty} I_m(f_m(s,\cdot)).$$

The kernels $f_m(s,t)$ can be chosen so that they belong to $L^2(T^{k+m})$ and are symmetric functions of $t \in T^m$ for all $s \in T^k$. We then can introduce the Skorohod integral δ^k of u.

DEFINITION 1.2.1 The multiple Skorohod integral of u is defined as ([4], [6])

$$\delta^k u = \sum_{m=0}^{\infty} I_{m+k}(\tilde{f}_m) \qquad (1.2.1)$$

where $\tilde{f}_m$ denotes the symmetrization of f_m over all its $(m+k)$ variables, provided that the series in (1.2.1) converges in $L^2(\Omega)$.

The domain of δ^k is the set

$$\text{Dom } \delta^k = \{u \in L^2(T^k \times \Omega): \sum_{m=0}^{\infty} (m+k)!\, \|\tilde{f}_m\|^2_{L^2(T^{m+k})} < \infty \}.$$

Clearly $\mathcal{H}^{(n)}(T^k \times \Omega) \subset \text{Dom } \delta^k$, $n = 1, 2, \dots$.

PROPOSITION 1.2.2 (Integration by parts formula) *Let* $F \in DomD^k$ *and* $u \in Dom\delta^k$, *then*

$$E(<D^k F, u>_{L^2(T^k)}) = E(F \delta^k u).$$

PROPOSITION 1.2.3. *For any* $p \in (1, \infty)$, *and* $u \in \mathcal{H}^{(k)}(T^k \times \Omega)$,

$$E(\,|\,\delta^k u\,|^p)$$
$$\leq C_{p,k} \{[E \int_{T^k} u_t^2\, \mu^k(dt)]^{p/2} + E\,|\int_{T^k} \int_{T^k} (D_s^k u_t)^2 \mu^k(dt) \mu^k(ds)\,|^{p/2}\}$$

PROPOSITION 1.2.4 (A formula of the Fubini type). *Let* $u \in \mathcal{H}^{(1)}(T^2 \times \Omega)$, *then*

$$\int \delta\,(u(s,\cdot))\mu(ds) = \delta\,(\int u(s,\cdot)\mu(ds)).$$

PROPOSITION 1.2.5 (A formula of the Fubini type) *Let* $u \in L^2(T^{k+j} \times \Omega)$ *and assume that*

(*a*) *For a.e.* $s \in T^k$, $u(s,\cdot) \in Dom\ \delta^j$,

(*b*) $\delta^j(u(s,\cdot)) \in Dom\ \delta^k$.

Then $u \in Dom\ \delta^{k+j}$ *and* $\delta^{k+j}(u) = \delta^k(\delta^j u)$.

PROPOSITION 1.2.6 . *Let* $u \in \mathcal{H}^{(2)}(T \times \Omega)$. *Then* $\delta u \in Dom\ D$ *and*

$$D_s(\delta u) = u_s + \int_T D_s u_r dW_r$$

for $s \in T$.

PROPOSITION 1.2.7. *Let* $F \in Dom\, D^k$, $u \in L^2(T^k \times \Omega)$ *satisfying*:

(*a*) *For all* $i = 1, \cdots, k$, $u \in Dom\delta^i$ *when any set of* $k-i$ *variables of* u *is fixed,*

(*b*) $(D^i_s F)\delta^{k-i}\tilde{u}(\cdot, s) \in L^2(T^i \times \Omega)$, *for all* $i = 0, \ldots, k$.

Then $Fu \in Dom\, \delta^k$ *and*

$$\delta^k(Fu) = F\delta^k u + \sum_{i=1}^{k} (-1)^i \binom{k}{i} \int_{T^i} (D^j_s F)\delta^{k-i}\tilde{u}(\cdot, s)\mu^i(ds).$$

PROPOSITION 1.2.8. *Let* $u \in \mathcal{H}^{(k)}(T^k \times \Omega)$ *and* $A \in \mathcal{F}$. *Suppose that*

$$u_t(\omega) = 0 \qquad d\mu^k(t) \times dP \quad a.e. \text{ on } \quad T^k \times A$$

Then $\delta^k u = 0$ *a.e. on* A.

Proposition 1.2.8 allows to localize the Skorohod integral as follows.

DEFINITION 1.2.9. Let $u \in L^2(T^k \times \Omega)$. We will say that u belongs to $\mathcal{H}^{(n)}_{loc}(T^k \times \Omega)$ whenever there exists a sequence $\Omega_j \uparrow \Omega$ a.s. and a sequence $\{u_j,\ j \in \mathbf{N}\} \subset \mathcal{H}^{(n)}(T^k \times \Omega)$ such that $u = u_j$ on Ω_j a.s. In that case we will say that u is localized by $\{(\Omega_j, u_j)\}$.

DEFINITION 1.2.10. Let $u \in \mathcal{H}^{(k)}_{loc}(T^k \times \Omega)$ be localized by $\{(\Omega_j,\ u_j)\}$. We then define $\delta^k u$ as the unique equivalence class of a.s. equal random variables such that

$$\delta^k u|_{\Omega_j} = \delta^k u_j|_{\Omega_j} \qquad \text{a.s.}$$

1.3. The Skorohod Integral as a Process in the Two-Parameter Case

Consider $T = [0, 1]^2$. $\mathcal{B}$ will be the Borel σ- field of T, and μ the Lebesgue measure on T. Let $z = (s, t)$ and $z' = (s', t')$ be two points in T. We write $z \leq z'$ if $s \leq s'$ and $t \leq t'$, $z < z'$ if $s < s'$ and $t < t'$, $z \,\overline{\wedge}\, z'$ if $s \leq s'$ and $t \geq t'$, $z \wedge z' = (\min(s, s'), \min(t, t'))$, $z \vee z' = (\max(s, s'), \max(t, t'))$. Furthermore, we denote by $z \otimes z' = (s, t')$, $[z, z']$ (resp. $(z, z']$) the rectangle $\{ r \in T, z \leq r \leq z' \}$ (resp. $\{ r \in T, z < r \leq z' \}$) and $R_z = [0, z]$. If $f : T \to \mathbb{R}$, $z = (s, t) \in T$, $z' = (s', t') \in T$ and $z < z'$, we denote by $f((z, z'])$ the increment of f on $(z, z']$, that means $f((z, z']) = f(z') - f(z \otimes z') - f(z' \otimes z) + f(z)$.

Let $\phi \in \mathcal{H}^{(1)}(T \times \Omega)$, $\psi \in \mathcal{H}^{(2)}(T^2 \times \Omega)$, $h \in \mathcal{H}^{(1)}(T^2 \times \Omega)$. Notice that $\phi \, 1_{R_z} \in \mathcal{H}^{(1)}(T \times \Omega)$, $\psi \, 1_{R_z \times R_z} \in \mathcal{H}^{(2)}(T^2 \times \Omega)$ and $h 1_{R_z \times R_z} \in \mathcal{H}^{(1)}(T^2 \times \Omega)$, we write

$$\delta \, (\phi \, 1_{R_z}) = \int_{R_z} \phi_\zeta dW_\zeta$$

$$\delta^2(\psi 1_{R_z \times R_z}) = \int_{R_z \times R_z} \psi(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$

$$\delta \, (\int_{R_z} h(\zeta, \cdot \,) d\zeta \, 1_{R_z}) = \int_{R_z \times R_z} h(\zeta, \zeta') d\zeta dW_{\zeta'}$$

PROPOSITION 1.3.1. *Let* $u \in \mathcal{H}^{(1)}(T \times \Omega)$. *For any two points* $z, z' \in T$ *with* $z < z'$, *we have*

$$E(\int_{(z, z']} u_\zeta dW_\zeta \mid \mathcal{F}_{(z, z']^c}) = 0.$$

PROPOSITION 1.3.2. *Let* $u \in \mathcal{H}^{(k)}(T^k \times \Omega)$. *If there exists some* $p > 2$ *such that*

$$\int_{T^k} (E u_t^2)^p dt < \infty$$

and

$$E \int_{T^k} (\int_{T^k} (D_s^k u_r)^2 \, ds)^p dr < \infty \, .$$

Then $\{ \delta^k(u 1_{(0, t]}) \equiv \int_{(0, t\,]} u_r d^k W_r \; , \; t \in T^k \}$ has a continuous version.

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Chapter 2

AN ITÔ FORMULA FOR TWO-PARAMETER SKOROHOD INTEGRALS

2.1. Introduction

In [5] Ustunel derived an Itô-type formula for anticipative stochastic processes of the form

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t \phi_s dW_s,$$

using the Malliavin calculus techniques and the fundamental theorem of the differential calculus. Applying Malliavin's redefinition technique [2], one can calculate the Lebesgue density $\frac{d}{dt}<F(X_t), \varphi>dt$ such that this derivative is continuous with respect to the parameter t, where φ is in a class of test functionals which spans Dom D. Then the Itô formula follows from the identity:

$$< F(X_t), \varphi > = < F(X_0), \varphi > + \int_0^t \frac{d}{ds} <F(X_s), \varphi>ds.$$

The aim of this chapter is to establish a generalized Itô formula for a two-parameter Skorohod integral process $X_z = (X_z^1, \dots, X_z^d)$ defined by the sum of multiple Skorohod integrals, a mixed Skorohod integral and ordinary Lebesgue integrals:

$$X_z^k = X_0^k + \int_{R_z} f_\zeta^k d\zeta + \int_{R_z} \phi_\zeta^k dW_\zeta + \int_{R_z \times R_z} \psi^k(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$
$$+ \int_{R_z \times R_z} h^k(\zeta, \zeta') d\zeta dW_{\zeta'} + \int_{R_z \times R_z} g^k(\zeta, \zeta') d\zeta d\zeta' \qquad (2.1.1)$$

Using Ustunel's approach, we get the one-parameter formula, Proposition 2.1.3, and then apply this result to derive Itô's formula for X_z

Jolis and Sanz [1] gave an Itô formula for $X_z = \int_{R_z} \phi_\zeta dW_\zeta$ by using the method as in the classical adapted case. Comparing with their result, the formula we obtain is under a rather weaker condition, i.e., we require $Du \in L^8(T^2\times\Omega)$, instead of $Du \in L^{8+\eta}(T^2\times\Omega)$, $\eta > 0$, in [1].

2.2. Lemmas

LEMMA 2.2.1. (Proposition 2.1 [4]) *Let* $F \in L^2(T^k\times\Omega)$ *and* $F^{(n)} \in \mathcal{H}^{(1)}(T^k\times\Omega)$ *for* $n = 1, 2, \ldots$. *Assume that* $F^{(n)} \to F$ *in* $L^2(T^k\times\Omega)$ *as* $n \to \infty$ *and that there is a function* $G \in L^2(T^{k+1}\times\Omega)$ *such that* $DF^{(n)} \to G$ *in* $L^2(T^{k+1}\times\Omega)$ *as* $n \to \infty$. *Then* $F \in \mathcal{H}^{(1)}(T^k\times\Omega)$, $DF = G$, *and* $F^{(n)} \to F$ *in* $\mathcal{H}^{(1)}(T^k\times\Omega)$, *as* $n \to \infty$.

Let (h_i) be a complete orthonormal system in $L^2(T)$ whose elements are smooth functions on T. For $k = 0, 1, \ldots$, let

$$C_k = \{ u \in L^2(T^k\times\Omega): u(t_1, \ldots, t_k) = f_1(t_1)\cdots f_k(t_k)\Phi(W(h_1), \ldots, W(h_m)),$$
$$f_i \in C_0^\infty(T),\ \Phi \in C_0^\infty(R^m), i = 1, \ldots, k,\ m = 1, 2, \ldots \} \qquad (2.2.1)$$

where $W(h) = \int_T h_\zeta dW_\zeta$. From the Proposition 2.7 of [4] or by the redefinition technique ([2], [5]), we have:

LEMMA 2.2.2. *The class* C_k *spans* $\mathcal{H}^{(n)}(T^k\times\Omega)$ for $n = 1, 2, \ldots$.

PROPOSITION 2.2.3. *Let* X_z^k , $z \in T$, $k = 1, 2, \ldots, d$ *be processes of the form* (2.1.1) *satisfying*

(*i*) *the process* $X_z = (X_z^1, \ldots, X_z^d)$ *has a continuous version,*

(*ii*) *for* $k = 0, 1, \cdots, d$, $X_0^k \in \mathcal{H}^{4,1}(\Omega)$, $f^k \in \mathcal{H}^{4,1}(T\times\Omega)$, $\phi^k \in \mathcal{H}^{4,2}(T\times\Omega)$,

$$\psi^k \in \mathcal{H}^{4,3}(T^2 \times \Omega), \; h^k \in \mathcal{H}^{4,2}(T^2 \times \Omega), \; g^k \in \mathcal{H}^{4,1}(T^2 \times \Omega).$$

Let $F \in C^2(\mathbf{R}^d)$, *then*

$$\begin{aligned} F(X_z) = F(X_0) &+ \sum_k \int_{R_z} F_k'(X_{\zeta \otimes z}) v^k(\zeta \otimes z; \zeta) d\zeta \\ &+ \sum_k \int_{R_z} F_k'(X_{\zeta \otimes z}) u^k(\zeta \otimes z; \zeta) dW_\zeta \\ &+ \frac{1}{2} \sum_{ij} \int_{R_z} F_{ij}''(X_{\zeta \otimes z}) u^i(\zeta \otimes z; \zeta) u^j(\zeta \otimes z; \zeta) d\zeta \\ &+ \sum_{ij} \int_{R_z} F_{ij}''(X_{\zeta \otimes z}) u^i(\zeta \otimes z; \zeta) D_\zeta^- X_{\zeta \otimes z}^j d\zeta \end{aligned} \tag{2.2.2}$$

where F_k', F_{ij}'' *denote partial derivatives, and* u, v, *and* $D^- X$ *are defined by*

$$u^k(z;\zeta) = \phi_\zeta^k + \int_{R_z} \tilde{\psi}^k(\zeta, \zeta') dW_{\zeta'} + \int_{R_z} h^k(\zeta', \zeta) d\zeta' \tag{2.2.3}$$

$$v^k(z;\zeta) = f_\zeta^k + \int_{R_z} h^k(\zeta, \zeta') dW_{\zeta'} + \int_{R_z} \tilde{g}^k(\zeta, \zeta') d\zeta' \tag{2.2.4}$$

$$\begin{aligned} D_\alpha^- X_z^k = D_\alpha X_0^k &+ \int_{R_z} D_\alpha f_\zeta^k d\zeta + \int_{R_z} D_\alpha \phi_\zeta^k dW_\zeta \\ &+ \int_{R_z \times R_z} D_\alpha \psi^k(\zeta, \zeta') dW_\zeta dW_{\zeta'} + \int_{R_z \times R_z} D_\alpha h^k(\zeta, \zeta') d\zeta \, dW_{\zeta'} \\ &+ \int_{R_z \times R_z} D_\alpha g^k(\zeta, \zeta') d\zeta d\zeta' \end{aligned} \tag{2.2.5}$$

where

$$\tilde{\psi}^k(\zeta, \zeta') = \psi^k(\zeta, \zeta') + \psi^k(\zeta', \zeta) \tag{2.2.6}$$

and

$$\tilde{g}^k(\zeta, \zeta') = g^k(\zeta, \zeta') + g^k(\zeta', \zeta). \tag{2.2.7}$$

Also

$$\begin{aligned} F(X_z) = F(X_0) &+ \sum_k \int_{R_z} F_k'(X_{z\otimes\zeta}) v^k(z\otimes\zeta; \zeta)\, d\zeta \\ &+ \sum_k \int_{R_z} F_k'(X_{z\otimes\zeta}) u^k(z\otimes\zeta; \zeta)\, dW_\zeta \\ &+ \frac{1}{2} \sum_{ij} \int_{R_z} F_{ij}''(X_{z\otimes\zeta}) u^i(z\otimes\zeta; \zeta) u^j(z\otimes\zeta; \zeta)\, d\zeta \\ &+ \sum_{ij} \int_R F_{ij}''(X_{z\otimes\zeta}) u^i(z\otimes\zeta; \zeta) D_\zeta^- X_{z\otimes\zeta}^j\, d\zeta \end{aligned} \tag{2.2.2)'$$

PROOF: To simplify the notation, we prove the proposition for $d = 1$. As in Ustunel [5] we first prove that the formula (2.2.2) holds if the integrands in (2.1.1) are smooth functions:

LEMMA 2.2.4. *Let* $F \in C_b^2(\mathbf{R}^d)$. *Suppose that* $X_0^k \in C_0$, $f^k, \phi^k \in C_1$, ψ^k, h^k, $g^k \in C_2$. *Then the formulas* (2.2.2) *and* (2.2.2)' *hold.*

PROOF: By the same argument as in the one-parameter case (Lemma 6.3, [4]), we will suppose that $F \in C_b^3(R)$. Let $z = (s, t)$ be a point of T. Using the Taylor formula, we have

$$F(X_{s+h,t}) - F(X_{s,t}) = F'(X_{s,t})(X_{s+h,t} - X_{s,t}) + \frac{1}{2} F''(\tilde{X}_{s,t})(X_{s+h,t} - X_{s,t})^2$$

where $\tilde{X}_{s,t}$ is an intermediate random point between $X_{s,t}$ and $X_{s+h,t}$. Then for $H \in C_0$,

$$\frac{1}{h} < F(X_{s+h,t}) - F(X_{s,t}),\ H >$$

$$= \frac{1}{h} < F'(X_{s,t})(X_{s+h,t} - X_{s,t}),\ H > + \ \frac{1}{2}\frac{1}{h} < F''(\widetilde{X}_{s,t})(X_{s+h,t} - X_{s,t})^2,\ H >$$

$$= I_1 + \frac{1}{2} I_2 , \quad \text{say.}$$

(1) ESTIMATION OF I_1 :

Write $\Delta_h = (s, s+h] \times [0, t]$, then $(0, s+h] \times [0, t] = R_z + \Delta_h$. From Fubini-type theorems of Propositions 1.2.4 and 1.2.5,

$$X_{s+h,t} - X_{s,t}$$

$$= \int_{\Delta_h} (f_\zeta d\zeta + \phi_\zeta dW_\zeta) + (\int_{R_z \times \Delta_h} + \int_{\Delta_h \times R_z} + \int_{\Delta_h \times \Delta_h})$$

$$(\psi(\zeta, \zeta')dW_\zeta dW_{\zeta'} + h(\zeta, \zeta')d\zeta dW_{\zeta'} + g(\zeta, \zeta')d\zeta d\zeta')$$

$$= \int_{\Delta_h} (f_\zeta + \int_{R_z} h(\zeta, \zeta')dW_{\zeta'} + \int_{R_z} \widetilde{g}(\zeta, \zeta')d\zeta')\, d\zeta$$

$$+ \int_{\Delta_h} (\phi_\zeta + \int_{R_z} \widetilde{\psi}(\zeta, \zeta')dW_{\zeta'} + \int_{R_z} h(\zeta', \zeta)d\zeta')\, dW_\zeta$$

$$+ \int_{\Delta_h \times \Delta_h} (\psi(\zeta, \zeta')dW_\zeta dW_{\zeta'} + h(\zeta, \zeta')d\zeta dW_{\zeta'} + g(\zeta, \zeta')d\zeta d\zeta')$$

$$= \int_{\Delta_h} v(z; \zeta)d\zeta + \int_{\Delta_h} u(z; \zeta)dW_\zeta$$

$$+ \int_{\Delta_h \times \Delta_h} (\psi(\zeta, \zeta')dW_\zeta dW_{\zeta'} + h(\zeta, \zeta')d\zeta dW_{\zeta'} + g(\zeta, \zeta')d\zeta d\zeta').$$

So

$$I_1 = \frac{1}{h} E[F'(X_z)H \int_{\Delta_h} v(z, \zeta)d\zeta\,]$$

$$+ \frac{1}{h} E[F'(X_z)H \int_{\Delta_h} u(z, \zeta)dW_\zeta\,]$$

$$+ \frac{1}{h} E[F'(X_z)H \int_{\Delta_h \times \Delta_h} (\psi(\zeta, \zeta')dW_\zeta dW_{\zeta'} + h(\zeta, \zeta')d\zeta dW_{\zeta'} + g(\zeta, \zeta')d\zeta d\zeta')]$$

$$= I_{11} + I_{12} + I_{13}\,, \quad \text{say.}$$

By the assumption,

$$I_{11} = \frac{1}{h} E[F'(X_z)H \int_s^{s+h} \int_0^t v(z; r_1, r_2) dr_2 dr_1]$$

$$\to \; E[F'(X_z)H \int_0^t v(z; s, r) dr] \tag{2.2.8}$$

as $h \to 0$. Indeed, let

$$f(\zeta) = f_1(\zeta)\Phi_1(W(h_1), \cdots, W(h_{m_1})),$$

$$h(\zeta, \zeta') = f_2(\zeta) f_3(\zeta')\Phi_2(W(h_1), \cdots, W(h_{m_2})),$$

$$g(\zeta, \zeta') = f_4(\zeta) f_5(\zeta')\Phi_3(W(h_1), \cdots, W(h_{m_3})),$$

where $f_i \in C_0^\infty(T)$, $\Phi_j \in C_0^\infty(R^{m_j})$, $i = 1, \ldots, 5$, $j = 1, 2, 3$. Write $\Phi_j = \Phi_j(W(h_1), \ldots, W(h_{m_j}))$. Then

$$\begin{aligned} v(z; \zeta) &= f_\zeta + \int_{R_z \times R_z} h(\zeta', \zeta) dW_{\zeta'} + \int_{R_z} \tilde{g}(\zeta, \zeta') d\zeta' \\ &= f_1(\zeta)\Phi_1 + f_2(\zeta) \int_{R_z} f_3(\zeta')\Phi_2 dW_{\zeta'} \\ &\quad + f_4(\zeta) \int_{R_z} f_5(\zeta')\Phi_3 d\zeta' + f_5(\zeta) \int_{R_z} f_4(\zeta')\Phi_3 d\zeta' \end{aligned}$$

So

$$\begin{aligned} I_{11} = E\{ F'(X_z)H[\Phi_1 \frac{1}{h}\int_s^{s+h} \int_0^t f_1(r_1, r_2) dr_2 dr_1 \\ + \int_{R_z} f_3(\zeta')\Phi_2 dW_{\zeta'} \frac{1}{h} \int_s^{s+h} \int_0^t f_2(r_1, r_2) dr_2 dr_1 \\ + \int_{R_z} f_5(\zeta')\Phi_3 d\zeta' \frac{1}{h} \int_s^{s+h} \int_0^t f_4(r_1, r_2) dr_2 dr_1 \end{aligned}$$

$$+ \int_{R_z} f_4(\zeta')\Phi_3 d\zeta' \; \frac{1}{h}\int_s^{s+h} \int_0^t f_5(r_1, r_2)dr_2dr_1]\}$$

$$\to \; E\{F'(X_z)H[\Phi_1\int_0^t f_1(s, r)dr + \int_{R_z} f_3(\zeta')\Phi_2 dW_{\zeta'} \int_0^t f_2(s, r)dr$$

$$+ \int_{R_z} f_5(\zeta')\Phi_3 d\zeta' \int_0^t f_4(s, r)dr \; + \int_{R_z} f_4(\zeta')\Phi_3 d\zeta' \int_0^t f_5(s, r)dr]\}$$

as $h \to 0$.

By Propositions 1.2.2, 1.1.4 and 1.2.7,

$$I_{12} = \frac{1}{h}E[\int_{\Delta_h} D_\zeta(F'(X_z)H\;)u(z, \zeta)d\zeta]$$

$$= \frac{1}{h}E[\int_{\Delta_h} F'(X_z)D_\zeta Hu(z, \zeta)d\zeta] \; + \; \frac{1}{h}E[\int_{\Delta_h} F''(X_z)Hu(z, \zeta)D_\zeta X_z d\zeta]$$

$$= \frac{1}{h}E[\int_{\Delta_h} F'(X_z)D_\zeta Hu(z, \zeta)d\zeta] \; + \; \frac{1}{h}E[\int_{\Delta_h} F''(X_z)Hu(z, \zeta)(D_\zeta X_0$$

$$+ \int_{R_z} D_\zeta f_\alpha d\alpha \; + \int_{R_z} D_\zeta \phi_\alpha dW_\alpha \; + \; \int_{R_z\times R_z} D_\zeta h(\alpha, \alpha')d\alpha dW_{\alpha'}$$

$$+ \int_{R_z\times R_z} D_\zeta \psi(\alpha, \alpha')dW_\alpha dW_{\alpha'} \; + \int_{R_z\times R_z} D_\zeta g(\alpha, \alpha')d\alpha d\alpha')d\zeta\,]$$

$$= \frac{1}{h}E[\int_{\Delta_h} F'(X_z)D_\zeta Hu(z, \zeta)d\zeta] \; + \; \frac{1}{h}E[\int_{\Delta_h} F''(X_z)Hu(z, \zeta)D_\zeta^- X_z d\zeta\,]$$

$$= I_{121} + I_{122}\,, \quad \text{say.}$$

Let

$$H = \Phi_0, X_0 = \Phi_1,\; f(\zeta) = f_1(\zeta)\Phi_2,\; \phi(\zeta) = f_2(\zeta)\Phi_3,$$

$$h(\zeta,\zeta') = f_3(\zeta)f_4(\zeta')\Phi_4,\; \psi(\zeta,\zeta') = f_5(\zeta)f_6(\zeta')\Phi_5,\; g(\zeta,\zeta') = f_7(\zeta)f_8(\zeta')\Phi_6\,,$$

where

$$f_i \in C_0^\infty(T), \quad \Phi_j = \Phi_j(W(h_1), \cdots, W(h_{m_j})), \quad \Phi_j \in C_0^\infty(R^{m_j}),$$

$i = 1, \cdots, 8$, $j = 0, 1, \cdots, 6$. Then

$$D_\zeta H = \sum_{i=1}^{m_0} \frac{\partial \Phi_0}{\partial x_i}(W(h_1), \cdots, W(h_{m_0})h_i(\zeta)$$

$$u(z;\zeta) = f_2(\zeta)\Phi_3 + f_5(\zeta)\int_{R_z} f_6(\zeta')\Phi_5 dW_{\zeta'}$$

$$+ f_6(\zeta)\int_{R_z} f_5(\zeta')\Phi_5 dW_{\zeta'} + f_4(\zeta)\int_{R_z} f_3(\zeta')\Phi_4 d\zeta'$$

and

$$D_\zeta^- X_z = D_\zeta\Phi_1 + D_\zeta\Phi_2\int_{R_z} f_1(\zeta')d\zeta' + \int_{R_z} D_\zeta\Phi_3 f_2(\zeta')dW_{\zeta'}$$

$$+ \int_{R_z\times R_z} D_\zeta\Phi_5 f_5(\zeta')f_6(\zeta'')dW_{\zeta'}dW_{\zeta''}$$

$$+ \int_{R_z\times R_z} D_\zeta\Phi_4 f_3(\zeta')f_4(\zeta'')d\zeta' dW_{\zeta''}$$

$$+ \int_{R_z\times R_z} D_\zeta\Phi_6 f_7(\zeta')f_8(\zeta'')d\zeta' d\zeta''.$$

Recall the smoothness of f and Φ we made, the convergences below follow from the Lebesgue dominated convergence theorem:

$$I_{121} = \frac{1}{h}E[\int_{\Delta_h} F'(X_z)D_\zeta Hu(z;\zeta)d\zeta]$$

$$\rightarrow E[F'(X_z)H\int_0^t u(z;s,r)D_{(s,r)}Hdr],$$

$$I_{122} = \frac{1}{h}E[\int_{\Delta_h} F''(X_z)Hu(z,\zeta)D_\zeta^- X_z d\zeta]$$

$$\rightarrow E[F''(X_z)H\int_0^t u(z;s,r)D_{(s,r)}^- X_z dr],$$

as $h \rightarrow 0$.

Hence

$$I_{12} \to E[F'(X_z)H\int_0^t u(z;s,r)D_{(s,r)}Hdr] + E[F''(X_z)H\int_0^t u(z;s,r)D^-_{(s,r)}X_z dr],$$

as $h \to 0$.

From Proposition 1.2.2,

$$I_{13} = \frac{1}{h}E[\int_{\Delta_h} D_{\zeta'}(F'(X_z)H)(\int_{\Delta_h}(\psi(\zeta,\zeta')dW_\zeta + h(\zeta,\zeta')d\zeta)d\zeta']$$

$$+ \frac{1}{h}E[F'(X_z)H\int_{\Delta_h \times \Delta_h} g(\zeta,\zeta')d\zeta d\zeta'].$$

Obviously the second term converges to zero as h tends to zero. Moreover the first term is bounded by

$$\text{const.}\,\frac{1}{h}\{E\int_{\Delta_h}[D_{\zeta'}(F'(X_z)H)]^2 d\zeta'\}^{1/2}$$

$$\times\{E\int_{\Delta_h}[\int_{\Delta_h}\psi(\zeta,\zeta')dW_\zeta]^2 d\zeta' + \int_{\Delta_h}\int_{\Delta_h} h(\zeta,\zeta')^2 d\zeta d\zeta'\}^{1/2}$$

$$\leq \text{const.}\, h^{1/2} \to 0,$$

as $h \to 0$. Here we have applied L^p- inequality of Proposition 1.2.3.

Therefore,

$$I_1 \to \int_0^t <F'(X_z)v(z;s,r), H>dr$$

$$+ \int_0^t <F'(X_z)u(z;s,r), D_{(s,r)}H>dr$$

$$+ \int_0^t <F''(X_z)u(z;s,r)D^-_{(s,r)}X_z, H>dr$$

as $h \to 0$.

(2) ESTIMATION OF I_2:

Write $\Delta_h X = X_{s+h,t} - X_{s,t}$. Then

$$
\begin{aligned}
&(X_{s+h,t} - X_{s,t})^2 \\
&\quad = \int_{\Delta_h} v(z;\zeta)d\zeta\, \Delta_h X + \int_{\Delta_h} u(z;\zeta)dW_\zeta \Delta_h X \\
&\quad + \int_{\Delta_h\times\Delta_h} [\psi(\zeta,\zeta')dW_\zeta dW_{\zeta'} + h(\zeta,\zeta')d\zeta dW_{\zeta'} + g(\zeta,\zeta')d\zeta d\zeta']\Delta_h X.
\end{aligned}
$$

So

$$
\begin{aligned}
I_2 &= \frac{1}{h} E[\, F''(\tilde{X}_{s,t})H\int_{\Delta_h} v(z;\zeta)\, d\zeta\, \Delta_h X] \\
&+ \frac{1}{h} E[F''(\tilde{X}_{s,t})H\int_{\Delta_h} u(z;\zeta)dW_\zeta \Delta_h X] \\
&+ \frac{1}{h} E[F''(\tilde{X}_{s,t})H\int_{\Delta_h\times\Delta_h} (\psi(\zeta,\zeta')dW_\zeta dW_{\zeta'} + h(\zeta,\zeta')d\zeta dW_{\zeta'} \\
&\qquad + g(\zeta,\zeta')d\zeta d\zeta')\Delta_h X] \\
&= I_{21} + I_{22} + I_{23}, \quad \text{say.}
\end{aligned}
$$

It is evident that

$$I_{21} \to 0,\ I_{23} \to 0, \quad \text{as} \quad h \to 0.$$

Next we will show

$$I_{22} \to \int_0^t <F''(X_z)u^2(z;s,r), H>dr$$

as $h \to 0$. Indeed from Propositions 1.2.2 and 1.2.7 we have

$$I_{22} = \frac{1}{h} E[\, F''(\tilde{X}_{s,t})H\int_{\Delta_h} \Delta_h X u(z;\zeta)dW_\zeta]$$

$$+ \ \frac{1}{h} E[\ F''(\tilde{X}_{s,t}) H \int_{\Delta_h} (D_\zeta \Delta_h X) u(z;\ \zeta) d\zeta\ ]$$

$$= \frac{1}{h} E[\ \int_{\Delta_h} D_\zeta (F''(\tilde{X}_{s,t}) H) \Delta_h X u(z;\ \zeta) d\zeta\ ]$$

$$+ \ \frac{1}{h} E[\ \int_{\Delta_h} F''(\tilde{X}_{s,t}) H (D_\zeta \Delta_h X) u(z;\ \zeta) d\zeta\ ]$$

$$= I_{221} + I_{222}\ , \text{ say.}$$

For fixed $\zeta \in \Delta_h$,

$$D_\zeta \Delta_h X = \int_{\Delta_h} D_\zeta v(z;\ \alpha) d\alpha \ + \int_{\Delta_h} D_\zeta u(z;\ \alpha) dW_\alpha$$
$$+ \int_{\Delta_h \times \Delta_h} D_\zeta \psi(\alpha,\ \alpha') dW_\alpha dW_{\alpha'} \ + \int_{\Delta_h} \tilde{\psi}(\zeta,\ \alpha) dW_\alpha$$
$$+ \int_{\Delta_h \times \Delta_h} D_\zeta h(\alpha,\ \alpha') d\alpha dW_{\alpha'} \ + \int_{\Delta_h} h(\alpha,\ \zeta) d\alpha \ + u(z;\ \zeta)$$
$$+ \int_{\Delta_h \times \Delta_h} D_\zeta g(\alpha,\ \alpha') d\alpha d\alpha'.$$

Using an argument similar to part (1), we can prove that

$$I_{222} \ \to E[F''(X_{s,t}) H \int_0^t u^2(z;\ s,\ r) dr]$$

as $h \to 0$. It is evident that I_{221} tends to zero when h goes to zero. Thus

$$I_{22} = \frac{1}{h} E[\ F''(\tilde{X}_{s,t}) H \int_{\Delta_h} u(z;\ \zeta) dW_\zeta \Delta_h X]$$
$$\to \int_0^t < F''(X_z) u^2(z;\ s,\ r),\ H > dr$$

as $h \to 0$.

Therefore,

$$I_2 \to \int_0^t < F''(X_z)u^2(z; s, r), H > dr$$

as $h \to 0$.

(3) PROOF OF THE LEMMA

Combining parts (1) and (2), we have proved that

$$\begin{aligned}
\frac{d<F(X_{r,t}), H>}{dr} \\
&= \int_0^t < F'(X_{r,t})v(r, t; r,\theta), H > d\theta \\
&+ \int_0^t < F'(X_{r,t})u(r, t; r,\theta), D_{(r,\theta)}H > d\theta \\
&+ \int_0^t < F''(X_{r,t})u(r, t; r,\theta)D^{-}_{(r,\theta)}X_{r,t}, H > d\theta \\
&+ \frac{1}{2} \int_0^t < F''(X_{r,t})u^2(r, t; r,\theta), H > d\theta
\end{aligned}$$

for any $H \in C_0$. It follows from the hypotheses that the derivative is continuous with respect to the parameter r. Applying the fundamental theorem of the differential calculus, we have

$$\begin{aligned}
<F(X_{s,t}), H> &= <F(X_{0,t}), H> + \int_0^s \frac{d<F(X_{r,t}), H>}{dr} dr \\
&= <F(X_0), H> + \int_0^s \int_0^t \{< F'(X_{r,t})v(r, t; r,\theta), H> \\
&+ < F''(X_{r,t})u(r, t; r,\theta)\, D^{-}_{(r,\theta)} X, H> + \frac{1}{2} < F''(X_{r,t})u^2(r, t; r,\theta), H> \\
&+ < F'(X_{r,t})u(r, t; r,\theta), D_{(r,\theta)}H>\} d\theta dr
\end{aligned}$$

$$
\begin{aligned}
&= <F(X_0), H> + \int_{R_z} <F'(X_{\zeta\otimes z})v(\zeta\otimes z;\, \zeta), H>d\zeta \\
&+ \int_{R_z} <F''(X_{\zeta\otimes z})u(\zeta\otimes z;\, \zeta)D^-_\zeta X_{\zeta\otimes z}, H>d\zeta \\
&+ \frac{1}{2} \int_{R_z} <F''(X_{\zeta\otimes z})u^2(\zeta\otimes z;\, \zeta), H>d\zeta \\
&+ \int_{R_z} <F'(X_{\zeta\otimes z})u(\zeta\otimes z;\, \zeta), D_\zeta H>d\zeta\, .
\end{aligned}
$$

Since δ is the adjoint of D,

$$\int_{R_z} <F'(X_{\zeta\otimes z})u(\zeta\otimes z;\, \zeta), D_\zeta H>d\zeta \;=\; <\int_{R_z} F'(X_{\zeta\otimes z})u(\zeta\otimes z;\, \zeta)dW_\zeta, H>.$$

Therefore for any $H \in C_0$, we have

$$
\begin{aligned}
&<F(X_z), H> - <F(X_0), H> \\
&\quad = < \int_{R_z} F'(X_{\zeta\otimes z})u(\zeta\otimes z;\, \zeta)dW_\zeta, H > + <\int_{R_z} F'(X_{\zeta\otimes z})v(\zeta\otimes z;\, \zeta)d\zeta, H > \\
&\quad + <\frac{1}{2} \int_{R_z} F''(X_{\zeta\otimes z})u^2(\zeta\otimes z;\, \zeta)d\zeta, H > \\
&\quad + <\int_{R_z} F'(X_{\zeta\otimes z})u(\zeta\otimes z;\, \zeta)D^-_\zeta X_{\zeta\otimes z}d\zeta, H >. \qquad (2.2.8)
\end{aligned}
$$

Denote by $\mathcal{L}_k$ the linear space generated by C_k, $k = 0, 1, \dots$. Clearly (2.2.8) holds for $H \in \mathcal{L}_0$. The lemma is proved since $\mathcal{L}_0$ is dense subset of $\mathcal{H}^{(1)}(\Omega)$.

PROOF OF PROPOSITION 2.2.3: From the assumption (ii), we have

$$X^k_z \in \mathcal{H}^{4,1}(\Omega),\ u^k(z;\, \cdot) \in \mathcal{H}^{4,2}(T\times\Omega),\ v^k(z;\, \cdot) \in \mathcal{H}^{4,1}(T\times\Omega),$$

for each $z \in T$. Indeed,

$$\psi \in \mathcal{H}^{4,3}(T^2\times\Omega) \Rightarrow \delta^2\psi \in \mathcal{H}^{4,1}(\Omega),\ \ \delta\psi(\zeta,\, \cdot) \in \mathcal{H}^{4,2}(T\times\Omega);$$

$$h \in \mathcal{H}^{4,2}(T^2\times\Omega) \Rightarrow \delta h(\zeta,\cdot) \in \mathcal{H}^{4,1}(T\times\Omega),\ \int_T h(\zeta',\zeta)d\zeta' \in \mathcal{H}^{4,2}(T\times\Omega),$$

$$\int_{T\times T} h(\zeta,\zeta')d\zeta dW_{\zeta'} \in \mathcal{H}^{4,1}(\Omega);$$

$$g \in \mathcal{H}^{4,1}(T^2\times\Omega) \Rightarrow \int_T g(\zeta,\zeta')d\zeta' \in \mathcal{H}^{4,1}(T\times\Omega);$$

$$f \in \mathcal{H}^{4,1}(T\times\Omega) \Rightarrow \int_T f_\zeta d\zeta \in \mathcal{H}^{4,1}(\Omega);$$

$$\phi \in \mathcal{H}^{4,2}(T\times\Omega) \Rightarrow \delta\phi \in \mathcal{H}^{4,1}(\Omega).$$

Therefore $F'(X_{\zeta\otimes z})u(\zeta\otimes z;\zeta) \in \mathcal{H}^{4,1}_{loc}(\Omega)$ and all other integrands in (2.2.2) belong to $L^2_{loc}(T\times\Omega)$, so that all terms of the formula (2.2.2) make sense. Note that X is continuous, by the localization arguments ([3]), it suffices to show the result for function F such that F and derivatives F', F'' are bounded. Following Lemma 2.2.2, we can choose a sequence $(X_0^n) \subset \mathcal{L}_0$ such that X_0^n converges to X_0 in $\mathcal{H}^{(1)}(\Omega)$. In fact, from the proof of the Lemma 2.2.2 ([3], [4]), we see that $X_0^n \to X_0$ in $L^4(\Omega)$ and $D X_0^n \to DX_0$ in $L^4(T\times\Omega)$. The similar procedures can be done for f, ϕ, ψ, h, g. Define

$$X_z^n = X_0^n + \int_{R_z} f_\zeta^n d\zeta + \int_{R_z} \phi_\zeta^n dW_\zeta + \int_{R_z\times R_z} \psi^n(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \int_{R_z\times R_z} h^n(\zeta,\zeta')d\zeta dW_{\zeta'} + \int_{R_z\times R_z} g^n(\zeta,\zeta')d\zeta d\zeta'$$

where $(f^n) \subset \mathcal{L}_1$, $(\phi^n) \subset \mathcal{L}_1$, $(\psi^n) \subset \mathcal{L}_2$, $(h^n) \subset \mathcal{L}_2$, $(g^n) \subset \mathcal{L}_2$, such that

(1) $f^n \to f,\ Df^n \to Df,\ g^n \to g,\ Dg^n \to Dg$

(2) $D^k\phi^n \to D^k\phi,\ k = 0, 1, 2$

(3) $D^k\psi^n \to D^k\psi,\ k = 0, 1, 2, 3$

(2) $D^k h^n \to D^k h, \quad k = 0, 1, 2,$

in L^4, as $n \to \infty$.

Then for each $z \in T$, we have

$$X_0^n \to X_0, \; X_z^n \to X_z, \; D^- X_z^n \to D^- X_z,$$

$$v^n(z; \cdot\,) \to v(z; \cdot\,), \; u^n(z; \cdot\,) \to u(z; \cdot\,), \; Du^n(z; \cdot\,) \to Du(z; \cdot\,)$$

in L^4.

From Lemma 2.2.4, the formula (2.2.2) holds for X^n, so does for X because of above convergences. The proof of the proposition is complete.

2.3. An Itô Formula

LEMMA 2.3.1. *Let* X *be of the form* (2.1.1) *satisfying* $f \in L^2(T\times\Omega)$, $g \in L^2(T^2\times\Omega)$, $\phi \in \mathcal{H}^{p,1}(T\times\Omega)$, $\psi \in \mathcal{H}^{p,2}(T^2\times\Omega)$, $h \in \mathcal{H}^{p,1}(T^2\times\Omega)$, *for some* $p > 4$. *Then the process* $\{X_z, z \in T\}$ *has a continuous version.*

PROOF. The Lemma follows immediately from the Proposition 1.3.2 by taking $t = (z, z)$ for $k = 2$.

THEOREM 2.3.2. *Let* $F \in C^4(\mathbf{R}^d)$ and X_z^k, $z \in T$, $k = 1, 2, \dots, d$ *be processes defined by* (2.1.1). Suppose:

$$X_0^k \in \mathcal{H}^{4,2}(\Omega), \; DX_0^k \in L^8(T\times\Omega),$$

$$f^k \in \mathcal{H}^{4,2}(T\times\Omega), \; Df^k \in L^8(T^2\times\Omega),$$

$$\phi^k \in \mathcal{H}^{8,2}(T\times\Omega) \cap \mathcal{H}^{4,3}(T\times\Omega),$$

$$\psi^k \in \mathcal{H}^{8,3}(T^2\times\Omega) \cap \mathcal{H}^{4,4}(T^2\times\Omega),$$

$$h^k \in \mathcal{H}^{8,2}(T^2\times\Omega) \cap \mathcal{H}^{4,3}(T^2\times\Omega),$$

$$g^k \in \mathcal{H}^{4,2}(T^2\times\Omega), \quad Dg^k \in L^8(T^3\times\Omega),$$

$k = 1, \cdots, d$. *Then*

$$\begin{aligned} F(X_z) = F(X_0) &+ \int_{R_z} S_1(\zeta)d\zeta + \int_{R_z} S_2(\zeta)dW_\zeta \\ &+ \int_{R_z\times R_z} I(\zeta'\,\overline{\wedge}\,\zeta)S_3(\zeta, \zeta')dW_\zeta dW_{\zeta'} + \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta, \zeta')S_4(\zeta, \zeta')d\zeta dW_{\zeta'} \\ &+ \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta, \zeta')S_5(\zeta, \zeta')d\zeta d\zeta' \end{aligned} \tag{2.3.1}$$

with

$$S_1(\zeta) = F'_k(X_\zeta)v^k(\zeta; \zeta) + \frac{1}{2}F''_{ij}(X_\zeta)u^i(\zeta; \zeta)u^j(\zeta; \zeta) + F''_{ij}(X_\zeta)u^i(\zeta; \zeta)D^-_\zeta X^j_\zeta \tag{2.3.2}$$

$$S_2(\zeta) = F'_k(X_\zeta)u^k(\zeta; \zeta) \tag{2.3.3}$$

$$S_3(\zeta, \zeta') = F''_{ij}(X_{\zeta\vee\zeta'})u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta') + F'_k(X_{\zeta\vee\zeta'})\tilde{\Psi}^k(\zeta, \zeta') \tag{2.3.4}$$

$$\begin{aligned} S_4(\zeta, \zeta') = F'_k(X_{\zeta\vee\zeta'})h^k(\zeta, \zeta') &+ F''_{ij}(X_{\zeta\vee\zeta'})[v^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta') \\ &+ \tilde{\Psi}^j(\zeta, \zeta')u^j(\zeta\vee\zeta'; \zeta) + u^i(\zeta\vee\zeta'; \zeta)D^-_\zeta u^j(\zeta\vee\zeta'; \zeta') + \tilde{\Psi}^i(\zeta, \zeta')D^-_\zeta X^j_{\zeta\vee\zeta'}] \\ &+ F^{(3)}_{ijk}(X_{\zeta\vee\zeta'})[\frac{1}{2}u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta)u^k(\zeta\vee\zeta'; \zeta') \\ &+ u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta')D^-_\zeta X^k_{\zeta\vee\zeta'}] \end{aligned} \tag{2.3.5}$$

$$\begin{aligned} S_5(\zeta, \zeta') = \frac{1}{2}F'_k(X_{\zeta\vee\zeta'})\tilde{g}^k(\zeta, \zeta') &+ F''_{ij}(X_{\zeta\vee\zeta'})[u^i(\zeta\vee\zeta'; \zeta')h^j(\zeta,\zeta') + \frac{1}{4}\tilde{\Psi}^i(\zeta, \zeta')\tilde{\Psi}^j(\zeta, \zeta') \\ &+ \frac{1}{2}v^i(\zeta\vee\zeta'; \zeta)v^j(\zeta\vee\zeta'; \zeta') + u^i(\zeta\vee\zeta'; \zeta)D^-_\zeta v^j(\zeta\vee\zeta'; \zeta') + h^i(\zeta,\zeta')D^-_{\zeta'} X^j_{\zeta\vee\zeta} \end{aligned}$$

$$
\begin{aligned}
&+ \tilde{\psi}^i(\zeta, \zeta')D^-_{\zeta}u^j(\zeta\vee\zeta'; \zeta') + \tfrac{1}{2}\tilde{\psi}^i(\zeta, \zeta')D^-_{\zeta}D^-_{\zeta'}X^j_{\zeta\vee\zeta'} \\
&+ \tfrac{1}{2}D^-_{\zeta}u^i(\zeta\vee\zeta'; \zeta')D^-_{\zeta'}u^j(\zeta\vee\zeta'; \zeta)] \\
&+ F^{(3)}_{ijk}(X_{\zeta\vee\zeta'})[\tfrac{1}{2}u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta)v^k(\zeta\vee\zeta'; \zeta') \\
&+ \tfrac{1}{2}u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta')\tilde{\psi}^k(\zeta, \zeta') + v^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta')D^-_{\zeta'}X^k_{\zeta\vee\zeta'} \\
&+ \tilde{\psi}^i(\zeta, \zeta')u^j(\zeta\vee\zeta'; \zeta)\,D^-_{\zeta'}X^k_{\zeta\vee\zeta'} + u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta')D^-_{\zeta'}u^k(\zeta\vee\zeta'; \zeta) \\
&+ u^i(\zeta\vee\zeta';\zeta)D^-_{\zeta}u^j(\zeta\vee\zeta';\zeta')D^-_{\zeta'}X^k_{\zeta\vee\zeta'} + \tfrac{1}{2}\tilde{\psi}^i(\zeta, \zeta')D^-_{\zeta}X^j_{\zeta\vee\zeta'}D^-_{\zeta'}X^k_{\zeta\vee\zeta'} \\
&+ \tfrac{1}{2}u^i(\zeta\vee\zeta';\zeta)u^j(\zeta\vee\zeta';\zeta')D^-_{\zeta}D^-_{\zeta'}X^k_{\zeta\vee\zeta'}\,] \\
&+ F^{(4)}_{ijkm}(X_{\zeta\vee\zeta'})[\tfrac{1}{8}u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta)u^k(\zeta\vee\zeta'; \zeta')u^m(\zeta\vee\zeta'; \zeta') \\
&+ \tfrac{1}{2}u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta)u^k(\zeta\vee\zeta'; \zeta')D^-_{\zeta'}X^m_{\zeta\vee\zeta'} \\
&+ \tfrac{1}{2}u^i(\zeta\vee\zeta'; \zeta)u^j(\zeta\vee\zeta'; \zeta')D^-_{\zeta}X^k_{\zeta\vee\zeta'}D^-_{\zeta'}X^m_{\zeta\vee\zeta'}\,]
\end{aligned}
\tag{2.3.6}
$$

where $I(\zeta\,\overline{\wedge}\,\zeta')$ *denotes the indicator function of the set* $\{\zeta\,\overline{\wedge}\,\zeta'\}$, $1_{\hat{T}^2}(\zeta, \zeta') = I(\zeta\,\overline{\wedge}\,\zeta') + I(\zeta'\,\overline{\wedge}\,\zeta)$, *and all repeated indices are summed from* 1 *to* d.

PROOF: As in adapted case [6], we will give expressions for every term of the integrands in (2.2.2) by applying (2.2.2)'. Fix $\zeta \in T$. For $z > \zeta$, we have

$$R_{\zeta\otimes z} \setminus R_{\zeta} = \{\alpha \in R_z : \alpha\,\overline{\wedge}\,\zeta\} = I(\cdot\,\overline{\wedge}\,\zeta) \cap R_z,$$

$$R_{\zeta\otimes z} \times R_{\zeta\otimes z} \setminus R_{\zeta} \times R_{\zeta}$$

$$= R_z \times R_z \cap \{I(\cdot \barwedge \zeta) \times I(\cdot \barwedge \zeta) + I(\cdot \barwedge \zeta) \times R_\zeta + R_\zeta \times I(\cdot \barwedge \zeta)\} \equiv R_z \times R_z \cap T_\zeta .$$

Thus for $1 \le k \le d$,

$$X^k_{\zeta \otimes z} = X^k_\zeta + \int_{R_z} I(\zeta' \barwedge \zeta)[\, f^k_{\zeta'} d\zeta' + \phi^k_{\zeta'} dW_{\zeta'}]$$

$$+ \int_{R_z \times R_z} 1_{T_\zeta}(\zeta', \zeta'')[\psi^k(\zeta', \zeta'')dW_{\zeta'}dW_{\zeta''}$$

$$+ \; h^k(\zeta', \zeta'')d\zeta' dW_{\zeta''} + g^k(\zeta', \zeta'')d\zeta' d\zeta'' \,]$$

$$u^k(\zeta \otimes z; \zeta) = u^k(\zeta; \zeta) + \int_{R_z} I(\zeta' \barwedge \zeta)[\,\tilde{\psi}^k(\zeta, \zeta')dW_{\zeta'} + \int_{R_z} h^k(\zeta', \zeta)d\zeta']$$

$$v^k(\zeta \otimes z; \zeta) = v^k(\zeta; \zeta) + \int_{R_z} I(\zeta' \barwedge \zeta)[h^k(\zeta, \zeta')dW_{\zeta'} + \tilde{g}^k(\zeta, \zeta')d\zeta']$$

$$D^-_\zeta X^k_{\zeta \otimes z} = D^-_\zeta X^k_\zeta + \int_{R_z} I(\zeta' \barwedge \zeta)[D_\zeta f^k_{\zeta'} d\zeta' + D_\zeta \phi^k_{\zeta'} dW_{\zeta'}]$$

$$+ \int_{R_z \times R_z} {}_{T_\zeta}(\zeta', \zeta'')[D_\zeta \psi^k(\zeta', \zeta'')dW_{\zeta'}dW_{\zeta''} + D_\zeta h^k(\zeta', \zeta'')d\zeta' dW_{\zeta''}$$

$$+ D_\zeta g^k(\zeta', \zeta'')d\zeta' d\zeta'']$$

Put

$$Y^i_z = X^i_{\zeta \otimes z}, \; i = 1, \dots, d, \;\; Y^{d+1}_z = u^k(\zeta \otimes z; \zeta).$$

Then $Y = (Y^1, \dots, Y^d, Y^{d+1})$ is a process with the form (2.1.1) and satisfies the condition (ii) of the Proposition 2.2.3. It follows from Lemma 2.2.1 that each process Y^i has a continuous version. Note that processes u, v corresponding to Y now are given by:

$$\bar{u}^i(z;\alpha) = X^i_\zeta + \int_{R_z} 1_{T\zeta}\tilde{\psi}^i(\alpha, \beta)dW_\beta + \int_{R_z} 1_{T\zeta}h^i(\beta, \alpha)d\beta,$$

$$\bar{u}^{d+1}(z;\alpha) = I(\alpha \bar{\wedge} \zeta)\tilde{\psi}^k(\zeta, \alpha),$$

$$\bar{v}^i(z;\alpha) = I(\alpha \bar{\wedge} \zeta)f^i_\zeta + \int_{R_z} 1_{T\zeta}h^i(\alpha, \beta)dW_\beta + \int_{R_z} 1_{T\zeta}\tilde{g}^i(\alpha, \beta)d\beta$$

$$\bar{v}^{d+1}(z;\alpha) = I(\alpha \bar{\wedge} \zeta)h^k(\alpha, \zeta),$$

$i = 1, \dots, d.$

Applying (2.2.2)' to $G(y^1, \dots, y^d, y^{d+1}) = F_k'(y^1, \dots, y^d)y^{d+1}$, we have

$$G(Y_z) = G(Y_0) + \sum_j \int_{R_z} G_j'(Y_{z\otimes\zeta'})[\bar{v}^j(z\otimes\zeta'; \zeta')d\zeta' + \bar{u}^j(z\otimes\zeta'; \zeta')dW_{\zeta'}]$$

$$+ \frac{1}{2}\sum_{ij} \int_{R_z} G_{ij}''(Y_{z\otimes\zeta'})\bar{u}^i(z\otimes\zeta'; \zeta')\bar{u}^j(z\otimes\zeta'; \zeta')d\zeta'$$

$$+ \sum_{ij} \int_{R_z} G_{ij}''(Y_{z\otimes\zeta})\bar{u}^i(z\otimes\zeta'; \zeta')D^-_{\zeta'}Y^j_{z\otimes\zeta'}\, d\zeta'$$

Note that $\zeta\otimes z\otimes\zeta' = \zeta\otimes\zeta'$, $1_{T\zeta}(\zeta', \beta)1_{R\zeta\otimes\zeta'}(\beta) = I(\zeta' \bar{\wedge} \zeta)1_{R\zeta\otimes\zeta'}(\beta)$. So

$$\bar{v}^j(z\otimes\zeta'; \zeta') = I(\zeta' \bar{\wedge} \zeta)v^j(\zeta\otimes\zeta'; \zeta'),$$

$$\bar{u}^j(z\otimes\zeta'; \zeta') = I(\zeta' \bar{\wedge} \zeta)u^j(\zeta\otimes\zeta'; \zeta'),$$

$$\bar{v}^{d+1}(z\otimes\zeta'; \zeta') = I(\zeta' \bar{\wedge} \zeta)h^k(\zeta', \zeta)$$

$$\bar{u}^{d+1}(z\otimes\zeta'; \zeta') = I(\zeta' \bar{\wedge} \zeta)\tilde{\psi}^k(\zeta, \zeta'),$$

$j = 1, 2, \dots, d.$ Thus

$$F_k'(X_{\zeta\otimes z})u^k(\zeta\otimes z; \zeta) = F_k'(X_\zeta)u^k(\zeta; \zeta)$$

$$+ \sum_{j} \int_{R_z} I(\zeta \barwedge \zeta) F''_{kj}(X_{\zeta\otimes\zeta'})[u^k(\zeta\otimes\zeta'; \zeta) v^j(\zeta\otimes\zeta'; \zeta') d\zeta'$$

$$+ \quad u^k(\zeta\otimes\zeta'; \zeta) u^j(\zeta\vee\zeta'; \zeta') dW_{\zeta'}]$$

$$+ \int_{R_z} I(\zeta \barwedge \zeta) F'_k(X_{\zeta\otimes\zeta'})[h^k(\zeta', \zeta) d\zeta' + \tilde{\psi}^k(\zeta, \zeta') dW_{\zeta'}]$$

$$+ \frac{1}{2} \sum_{ij} \int_{R_z} I(\zeta \barwedge \zeta) F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'}) u^k(\zeta\otimes\zeta'; \zeta) u^j(\zeta\otimes\zeta'; \zeta') u^i(\zeta\otimes\zeta'; \zeta') d\zeta'$$

$$+ \sum_{j} \int_{R_z} I(\zeta \barwedge \zeta) F''_{kj}(X_{\zeta\otimes\zeta'}) u^j(\zeta\otimes\zeta'; \zeta') \tilde{\psi}^k(\zeta, \zeta') d\zeta'$$

$$+ \sum_{ij} \int_{R_z} I(\zeta \barwedge \zeta) F^{(3)}_{ijk}(\mathrm{X}_{\zeta\otimes\zeta'}) u^k(\zeta\otimes\zeta'; \zeta) u^i(\zeta\otimes\zeta'; \zeta') D^{-}_{\zeta'} X^j_{\zeta\vee\zeta} \, d\zeta'$$

$$+ \sum_{j} \int_{R_z} I(\zeta \barwedge \zeta) F''_{kj}(X_{\zeta\otimes\zeta'}) u^j(\zeta\otimes\zeta'; \zeta') D^{-}_{\zeta'} u^k(\zeta\vee\zeta'; \zeta) d\zeta'$$

$$+ \sum_{j} \int_{R_z} I(\zeta \barwedge \zeta) F''_{kj}(X_{\zeta\otimes\zeta'}) \tilde{\psi}^k(\zeta, \zeta') D^{-}_{\zeta'} X^j_{\zeta\vee\zeta} \, d\zeta' \qquad (2.3.7)$$

By hypotheses

$$DX \in L^8(T^2\times\Omega),\ D^2X \in L^4(T^3\times\Omega),\ u \in \mathcal{H}^{8,1}(T^2\times\Omega)\cap\mathcal{H}^{4,2}(T^2\times\Omega).$$

So

$$F''_{kj}(\mathrm{X}_{\zeta\otimes\zeta'}) u^k(\zeta\otimes\zeta'; \zeta) u^j(\zeta\vee\zeta'; \zeta'),\ F'_k(\mathrm{X}_{\zeta\otimes\zeta'}) \tilde{\psi}^k(\zeta, \zeta') \in \mathcal{H}^{(2)}_{loc}(T^2\times\Omega),$$

and all other terms of integrands in (2.3.7) belong to $\mathcal{H}^{(1)}_{loc}(T^2\times\Omega)$. From Propositions 1.2.5 and 1.2.6 , we have

$$\int_{R_z} F_k'(X_{\zeta\otimes z})u^k(\zeta\otimes z;\, \zeta)dW_\zeta = \int_{R_z} F_k'(X_\zeta)u^k(\zeta;\, \zeta)dW_\zeta$$

$$+ \sum_j \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{kj}''(X_{\zeta\otimes\zeta'})[u^k(\zeta\otimes\zeta';\, \zeta)v^j(\zeta\otimes\zeta';\, \zeta')d\zeta' dW_\zeta$$

$$+ u^k(\zeta\otimes\zeta';\zeta)u^j(\zeta\vee\zeta';\, \zeta')dW_{\zeta'} dW_\zeta]$$

$$+ \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_k'(X_{\zeta\otimes\zeta'})[h^k(\zeta',\, \zeta)d\zeta' dW_\zeta + \tilde{\psi}^k(\zeta,\, \zeta')dW_{\zeta'} dW_\zeta]$$

$$+ \frac{1}{2}\sum_{ij} \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{ijk}^{(3)}(X_{\zeta\otimes\zeta'})u^k(\zeta\otimes\zeta';\, \zeta)u^j(\zeta\otimes\zeta';\, \zeta')u^i(\zeta\otimes\zeta';\, \zeta')d\zeta' dW_\zeta$$

$$+ \sum_j \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{kj}''(X_{\zeta\otimes\zeta'})u^j(\zeta\otimes\zeta';\, \zeta')\tilde{\psi}^k(\zeta,\, \zeta')d\zeta' dW_\zeta$$

$$+ \sum_{ij} \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{ijk}^{(3)}(X_{\zeta\otimes\zeta'})u^k(\zeta\otimes\zeta';\, \zeta)u^i(\zeta\otimes\zeta';\, \zeta')D_{\zeta'}^{-} X_{\zeta\vee\zeta'}^{j}\, d\zeta' dW_\zeta$$

$$+ \sum_j \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{kj}''(X_{\zeta\otimes\zeta'})u^j(\zeta\otimes\zeta';\, \zeta')D_{\zeta'}^{-} u^k(\zeta\vee\zeta';\, \zeta)d\zeta' dW_\zeta$$

$$+ \sum_j \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{kj}''(X_{\zeta\otimes\zeta'})\tilde{\psi}^k(\zeta,\, \zeta')D_{\zeta'}^{-} X_{\zeta\vee\zeta'}^{j}\, d\zeta' dW_\zeta.$$

Similarly, letting $Y_z^{d+1} = v^k(\zeta\otimes z;\, \zeta)$, we have

$$\int_{R_z} F_k'(X_{\zeta\otimes z})v^k(\zeta\otimes z;\, \zeta)d\zeta$$

$$= \int_{R_z} F_k'(X_\zeta)v^k(\zeta;\, \zeta)d\zeta$$

$$+ \sum_j \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)F_{kj}''(X_{\zeta\otimes\zeta'})[v^k(\zeta\otimes\zeta';\, \zeta)v^j(\zeta\otimes\zeta';\, \zeta')d\zeta' d\zeta$$

$$+ v^k(\zeta\otimes\zeta';\zeta)u^j(\zeta\vee\zeta';\, \zeta')dW_{\zeta'} d\zeta\,]$$

$$+ \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) F'_k(X_{\zeta \otimes \zeta'})[h^k(\zeta, \zeta') dW_{\zeta'} d\zeta + \tilde{g}^k(\zeta', \zeta) d\zeta' d\zeta]$$

$$+ \frac{1}{2} \sum_{ij} \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) F^{(3)}_{ijk}(X_{\zeta \otimes \zeta'}) v^k(\zeta \otimes \zeta'; \zeta) u^j(\zeta \otimes \zeta'; \zeta') u^i(\zeta \otimes \zeta'; \zeta') d\zeta' d\zeta$$

$$+ \sum_{j} \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) F''_{kj}(X_{\zeta \otimes \zeta'}) u^j(\zeta \otimes \zeta'; \zeta') h^k(\zeta, \zeta') d\zeta' d\zeta$$

$$+ \sum_{ij} \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) F^{(3)}_{ijk}(X_{\zeta \otimes \zeta'}) v^k(\zeta \otimes \zeta'; \zeta) u^i(\zeta \otimes \zeta'; \zeta') D^{-}_{\zeta'} X^j_{\zeta \vee \zeta'}\, d\zeta' d\zeta$$

$$+ \sum_{j} \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) F''_{kj}(X_{\zeta \otimes \zeta'}) u^j(\zeta \otimes \zeta'; \zeta') D^{-}_{\zeta'} v^k(\zeta \vee \zeta'; \zeta) d\zeta' d\zeta$$

$$+ \sum_{j} \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) F''_{kj}(X_{\zeta \otimes \zeta'}) h^k(\zeta, \zeta') D^{-}_{\zeta'} X^j_{\zeta \vee \zeta'}\, d\zeta' d\zeta \,. \tag{2.3.8}$$

For $1 \leq k, j \leq d$, let

$$Y^{d+1}_z = u^k(\zeta \otimes z; \zeta), \quad Y^{d+2}_z = u^k(\zeta \otimes z; \zeta), \quad \text{or} \quad Y^{d+2}_z = D^{-}_{\zeta} X^j_{\zeta \otimes z}\,,$$

and

$$G(y^1, \cdots, y^d, y^{d+1}, y^{d+2}) = F''_{ij}(y^1, \cdots, y^d) y^{d+1} y^{d+2} \,.$$

Then

$$\frac{1}{2} \int_{R_z} F''_{ij}(X_{\zeta \otimes z}) u^i(\zeta \otimes z; \zeta) u^j(\zeta \otimes z; \zeta) d\zeta = \frac{1}{2} \int_{R_z} F''_{ij}(X_{\zeta}) u^i(\zeta; \zeta) u^j(\zeta; \zeta) d\zeta$$

$$+ \frac{1}{2} \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) \{ F^{(3)}_{ijk}(X_{\zeta \otimes \zeta'}) u^i(\zeta \otimes \zeta'; \zeta) u^j(\zeta \otimes \zeta'; \zeta) v^k(\zeta \otimes \zeta'; \zeta') d\zeta' d\zeta$$

$$+ F''_{ij}(X_{\zeta \otimes \zeta'})[u^i(\zeta \otimes \zeta'; \zeta) h^j(\zeta', \zeta) + u^j(\zeta \otimes \zeta'; \zeta) h^i(\zeta', \zeta')] d\zeta' d\zeta$$

$$+ F^{(3)}_{ijk}(X_{\zeta \otimes \zeta'}) u^i(\zeta \vee \zeta'; \zeta) u^j(\zeta \otimes \zeta'; \zeta) u^k(\zeta \otimes \zeta'; \zeta') dW_{\zeta'} d\zeta$$

$$+ \; F''_{ij}(X_{\zeta\otimes\zeta'})[\tilde{\psi}^{i}(\zeta, \zeta')u^{j}(\zeta\otimes\zeta'; \zeta) + \tilde{\psi}^{j}(\zeta, \zeta')u^{i}(\zeta\otimes\zeta'; \zeta)]dW_{\zeta'}d\zeta$$

$$+ \; \frac{1}{2}F^{(4)}_{ijkm}(X_{\zeta\otimes\zeta'})u^{i}(\zeta\otimes\zeta'; \zeta)u^{j}(\zeta\otimes\zeta'; \zeta)u^{k}(\zeta\otimes\zeta'; \zeta')u^{m}(\zeta\otimes\zeta'; \zeta')d\zeta'd\zeta$$

$$+ \; F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'})[u^{i}(\zeta\otimes\zeta'; \zeta)u^{k}(\zeta\otimes\zeta'; \zeta')\tilde{\psi}^{j}(\zeta, \zeta')$$

$$+ \; u^{j}(\zeta\otimes\zeta';\zeta)u^{k}(\zeta\otimes\zeta'; \zeta')\tilde{\psi}^{i}(\zeta, \zeta')]d\zeta'd\zeta$$

$$+ \; F''_{ij}(X_{\zeta\otimes\zeta'})\tilde{\psi}^{i}(\zeta, \zeta')\tilde{\psi}^{j}(\zeta, \zeta')d\zeta'd\zeta$$

$$+ \; F^{(4)}_{ijkm}(X_{\zeta\otimes\zeta'})u^{i}(\zeta\otimes\zeta'; \zeta)u^{j}(\zeta\otimes\zeta'; \zeta)u^{k}(\zeta\otimes\zeta'; \zeta')D^{-}_{\zeta'}X^{m}_{\zeta\otimes\zeta'}\,d\zeta'd\zeta$$

$$+ \; F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'})[u^{i}(\zeta\otimes\zeta'; \zeta)u^{k}(\zeta\otimes\zeta'; \zeta')D^{-}_{\zeta'}u^{j}(\zeta\otimes\zeta'; \zeta)$$

$$+ \; u^{j}(\zeta\otimes\zeta'; \zeta)u^{k}(\zeta\otimes\zeta'; \zeta')D^{-}_{\zeta'}u^{i}(\zeta\otimes\zeta'; \zeta) + \tilde{\psi}^{i}(\zeta, \zeta')u^{j}(\zeta\otimes\zeta'; \zeta)\,D^{-}_{\zeta'}X^{k}_{\zeta\otimes\zeta'}$$

$$+ \; \tilde{\psi}^{j}(\zeta, \zeta')u^{i}(\zeta\otimes\zeta'; \zeta)\,D^{-}_{\zeta'}X^{k}_{\zeta\otimes\zeta'}\,]d\zeta'd\zeta \; + \; F''_{ij}(X_{\zeta\otimes\zeta'})[\tilde{\psi}^{i}(\zeta, \zeta')D^{-}_{\zeta}u^{j}(\zeta\otimes\zeta';\zeta')$$

$$+ \; \tilde{\psi}^{j}(\zeta, \zeta')D^{-}_{\zeta}u^{i}(\zeta\otimes\zeta';\zeta')]d\zeta'd\zeta\} \qquad (2.3.9)$$

where indices *k*, *m* are summed from 1 to *d*.

Also

$$\int_{R_z} F''_{ij}(X_{\zeta\otimes z})u^{i}(\zeta\otimes z; \zeta)D^{-}_{\zeta}X^{j}_{\zeta\otimes z}d\zeta = \int_{R_z} F''_{ij}(X_{\zeta})u^{i}(\zeta; \zeta)D^{-}_{\zeta}\,X^{j}_{\zeta}\,d\zeta$$

$$+ \int_{R_z\times R_z} I(\zeta'\overline{\wedge}\zeta)\{F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'})u^{i}(\zeta\otimes\zeta'; \zeta)v^{k}(\zeta\otimes\zeta'; \zeta')D^{-}_{\zeta}\,X^{j}_{\zeta\otimes\zeta'}\,d\zeta'd\zeta$$

$$+ \; F''_{ij}(X_{\zeta\otimes\zeta'})[h^{i}(\zeta',\zeta)D^{-}_{\zeta}X^{j}_{\zeta\otimes\zeta'} \; + \; u^{i}(\zeta\otimes\zeta';\zeta)D^{-}_{\zeta}v^{j}(\zeta\otimes\zeta'; \zeta')]d\zeta'd\zeta$$

$$
\begin{aligned}
&+ F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'})u^i(\zeta\otimes\zeta';\zeta)u^k(\zeta\otimes\zeta';\zeta')D^{-}_{\zeta}X^j_{\zeta\otimes\zeta'}\,dW_{\zeta'}d\zeta \\
&+ F''_{ij}(X_{\zeta\otimes\zeta'})[\tilde{\psi}^i(\zeta,\zeta')D^{-}_{\zeta}X^j_{\zeta\otimes\zeta'} + u^i(\zeta\otimes\zeta';\zeta)D^{-}_{\zeta}u^j(\zeta\otimes\zeta';\zeta')]dW_{\zeta'}d\zeta \\
&+ \frac{1}{2}F^{(4)}_{ijkm}(X_{\zeta\otimes\zeta'})u^i(\zeta\otimes\zeta';\zeta)u^k(\zeta\otimes\zeta';\zeta')u^m(\zeta\otimes\zeta';\zeta')D_{\zeta}X^j_{\zeta\otimes\zeta'}\,d\zeta'd\zeta \\
&+ F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'})[\tilde{\psi}^i(\zeta,\zeta')u^k(\zeta\otimes\zeta';\zeta')\,D^{-}_{\zeta}X^j_{\zeta\otimes\zeta} \\
&+ u^i(\zeta\otimes\zeta';\zeta)u^k(\zeta\otimes\zeta';\zeta')D^{-}_{\zeta}u^j(\zeta\otimes\zeta';\zeta')]d\zeta'd\zeta \\
&+ \frac{1}{2}F''_{ij}(X_{\zeta\otimes\zeta'})[\tilde{\psi}^i(\zeta,\zeta')D^{-}_{\zeta}u^j(\zeta\otimes\zeta';\zeta')d\zeta'd\zeta + \tilde{\psi}^j(\zeta,\zeta')D^{-}_{\zeta}u^i(\zeta\otimes\zeta';\zeta')d\zeta'd\zeta\,] \\
&+ F^{(4)}_{ijkm}(X_{\zeta\otimes\zeta'})u^i(\zeta\otimes\zeta';\zeta)u^k(\zeta\otimes\zeta';\zeta')D^{-}_{\zeta}X^j_{\zeta\otimes\zeta'}\,D^{-}_{\zeta'}X^m_{\zeta\otimes\zeta'}\,d\zeta'd\zeta \\
&+ F^{(3)}_{ijk}(X_{\zeta\otimes\zeta'})[u^k(\zeta\otimes\zeta';\zeta')D^{-}_{\zeta'}u^i(\zeta\otimes\zeta';\zeta)D^{-}_{\zeta}X^j_{\zeta\otimes\zeta'} \\
&+ \tilde{\psi}^i(\zeta,\zeta')D^{-}_{\zeta}X^j_{\zeta\otimes\zeta'}\,D^{-}_{\zeta'}X^k_{\zeta\otimes\zeta'} + u^i(\zeta\otimes\zeta';\zeta)u^k(\zeta\otimes\zeta';\zeta')D^{-}_{\zeta}D^{-}_{\zeta'}X^j_{\zeta\otimes\zeta'} \\
&+ u^i(\zeta\otimes\zeta';\zeta)D^{-}_{\zeta}u^j(\zeta\otimes\zeta';\zeta')D^{-}_{\zeta'}X^k_{\zeta\otimes\zeta'}\,]d\zeta'd\zeta \\
&+ F''_{ij}(X_{\zeta\otimes\zeta'})[\tilde{\psi}^i(\zeta,\zeta')D^{-}_{\zeta}D^{-}_{\zeta'}X^j_{\zeta\otimes\zeta} + D^{-}_{\zeta'}u^i(\zeta\otimes\zeta';\zeta)D^{-}_{\zeta}u^j(\zeta\otimes\zeta';\zeta')]d\zeta'd\zeta
\end{aligned}
\qquad (2.3.10)
$$

where indices k, m are summed from 1 to d.

Notice that on $I(\zeta'\barwedge\zeta)$, $\zeta\otimes\zeta' = \zeta\vee\zeta'$. The proof of the theorem is completed.

REMARK 2.3.3. If integrands in (2.1.1) are nonanticipating, all terms with derivative D are zero. Thus (2.3.1) becomes the Itô formula given by Wong and Zakai [5].

COROLLARY 2.3.4. *Let* $F \in C^4(\mathbb{R}^d)$ *and* $X_z = (X_z^1, \dots, X_z^d)$ *be of the form*

$$X_z^k = X_0^k + \int_{R_z} f_\zeta^k d\zeta + \int_{R_z} \phi_\zeta^k dW_\zeta \tag{2.3.11}$$

Suppose

$$X_0^k \in \mathcal{H}^{4,2}(\Omega),\ DX_0^k \in L^8(T\times\Omega),$$

$$f^k \in \mathcal{H}^{4,2}(T\times\Omega),\ Df^k \in L^8(T^2\times\Omega),$$

$$\phi^k \in \mathcal{H}^{8,2}(T\times\Omega)\cap\mathcal{H}^{4,3}(T\times\Omega),$$

then $F(X_z)$ *is of the form* (2.3.1) *with*

$$S_1(\zeta) = F_k'(X_\zeta) f_\zeta^k + \frac{1}{2} F_{ij}''(X_\zeta)\phi_\zeta^i \phi_\zeta^j + F_{ij}''(X_\zeta)\phi_\zeta^i D_\zeta^- X_\zeta^j \tag{2 .3.12}$$

$$S_2(\zeta) = F_k'(X_\zeta)\phi_\zeta^k \tag{2.3.13}$$

$$S_3(\zeta, \zeta') = F_{ij}''(X_{\zeta\vee\zeta'})\phi_\zeta^i \phi_{\zeta'}^j \tag{2.3.14}$$

$$S_4(\zeta, \zeta') = F_{ij}''(X_{\zeta\vee\zeta'})[f_\zeta^i \phi_{\zeta'}^j + \phi_\zeta^i D_\zeta \phi_{\zeta'}^j]$$

$$+ F_{ijk}^{(3)}(X_{\zeta\vee\zeta'})[\frac{1}{2}\phi_\zeta^i \phi_\zeta^j \phi_{\zeta'}^k + \phi_\zeta^i \phi_{\zeta'}^j D_\zeta^- X_{\zeta\vee\zeta'}^k] \tag{2.3.15}$$

$$S_5(\zeta, \zeta') = F_{ij}''(X_{\zeta\vee\zeta'})[\frac{1}{2} f_\zeta^i f_{\zeta'}^j + \phi_\zeta^i D_\zeta f_{\zeta'}^j + \frac{1}{2} D_\zeta \phi_{\zeta'}^i D_{\zeta'} \phi_\zeta^j]$$

$$+ F_{ijk}^{(3)}(X_{\zeta\vee\zeta'})[\frac{1}{2}\phi_\zeta^i \phi_\zeta^j f_{\zeta'}^k + f_\zeta \phi_{\zeta'}^j D_{\zeta'}^- X_{\zeta\vee\zeta'}^k + \phi_\zeta^i D_\zeta \phi_{\zeta'}^j D_{\zeta'}^- X_{\zeta\vee\zeta'}^k$$

$$+ \ \phi^i_\zeta \phi^j_{\zeta'} D_{\zeta'} \phi^k_\zeta + \frac{1}{2} \phi^i_\zeta \phi^j_{\zeta'} D^-_{\zeta'} D^-_\zeta X^k_{\zeta\vee\zeta'} \ ]$$

$$+ \ F^{(4)}_{ijkm}(X_{\zeta\vee\zeta'})[\ \frac{1}{8} \phi^i_\zeta \phi^j_\zeta \phi^k_{\zeta'} \phi^m_{\zeta'} + \frac{1}{2} \phi^i_\zeta \phi^j_\zeta \phi^k_{\zeta'} D^-_{\zeta'} X^m_{\zeta\vee\zeta}$$

$$+ \ \frac{1}{2} \phi^i_\zeta \phi^j_{\zeta'} D^-_\zeta X^k_{\zeta\vee\zeta} D^-_{\zeta'} X^m_{\zeta\vee\zeta'} \ ]. \tag{2.3.16}$$

COROLLARY 2.3.5. *The conclusion of Corollary* 2.3.4 *remains valid if we replace the hypotheses concerning* (X^k_0, f^k, ϕ^k) *by*

(*i*) $X_z = (X^1_z , \ldots , X^d_z)$ *has a continuous version,*

(*ii*) $X^k_0 \in \mathcal{H}^{4,1}(\Omega)\cap\mathcal{H}^{2,2}(\Omega)$, $f^k \in \mathcal{H}^{4,1}(T\times\Omega)\cap\mathcal{H}^{2,2}(T\times\Omega)$,

$\phi^k \in \mathcal{H}^{4,2}(T\times\Omega)\cap\mathcal{H}^{2,3}(T\times\Omega)\cap L^\infty(T\times\Omega)$.

Moreover, we may drop the requirement (*i*), *provided we assume that* $F \in C^4_b(R^d)$.

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Chapter 3

AN ITÔ FORMULA FOR TWO-PARAMETER SKOROHOD INTEGRALS IN THE STRATONOVICH FORM

3.1 Introduction

The Skorohod-Stratonovich integral in the one-parameter case was introduced and investigated by Nualart and Pardoux [2]. In this chapter we consider the analog of the two-parameter Skorohod-Stratonovich integral. The Itô formula in the Stratonovich form is obtained, which formally agrees with the ordinary rule of calculus.

Section 3.2 introduces the stochastic processes $X*Y$, $X\circ Y$, $[X, Y]$, $\langle X, Y\rangle_i$, and $\langle\!\!\!-\, X, Y\rangle_i$. Their integral representations are given in Section 3.3. The Itô formula (2.3.1) with new notations is contained in Section 3.4. In Section 3.5 we establish the Itô formula in the Stratonovich form.

3.2 Definitions

In the sequel we use the notations of Hajek [1]. Let

$$\begin{aligned} X_z = X_0 &+ \int_{R_z} f_\zeta d\zeta + \int_{R_z} \phi_\zeta dW_\zeta + \int_{R_z\times R_z} \psi(\zeta, \zeta')dW_\zeta dW_{\zeta'} \\ &+ \int_{R_z\times R_z} h(\zeta, \zeta')d\zeta dW_{\zeta'} + \int_{R_z\times R_z} g(\zeta, \zeta')d\zeta d\zeta' \end{aligned} \tag{3.2.1}$$

as in Chapter 2.

For integers $n \geq 1$, $k \geq 0$ and $p \in [2, \infty)$, let $\mathcal{S}^{p,n}$ be the linear space of processes of the form (3.2.1), where

$$X_0 \in \mathcal{H}^{p,n-1}(\Omega),\ f \in \mathcal{H}^{p,n-1}(T\times\Omega),\ g \in \mathcal{H}^{p,n-1}(T^2\times\Omega),\ \phi \in \mathcal{H}^{p,n}(T\times\Omega),$$

$$h \in \mathcal{H}^{p,n}(T^2\times\Omega),\ \psi \in \mathcal{H}^{p,n+1}(T^2\times\Omega).$$

Here $\mathcal{H}^{p,0}(T^k\times\Omega)$ stands for $L^p(T^k\times\Omega)$.

For $X \in \mathcal{S}^{2,1}$, we define $u(z;\zeta)$, $v(z;\zeta)$ and $D^-_\zeta X_z$ by (2.2.3), (2.2.4) and (2.2.5) respectively. It follows from the hypothesis that u, v, D^-X make sense and belong to $L^2(T^2\times\Omega)$.

Let $[0, z] = [0, s]\times[0, t]$, $z = (s, t) \in T$, be a fixed rectangle; $0 = s_0 < s_1 < \cdots < s_m = s$ and $0 = t_0 < t_1 < \cdots < t_n = t$ be arbitrary partitions of the intervals $[0, s]$ and $[0, t]$ respectively; $z_{kl} = (s_k, t_l)$; $\|\sigma\| = \max_{kl}\{(s_{k+1} - s_k), (t_{l+1} - t_l)\}$. For any function f on T, we denote by $f_{kl} = f(z_{kl})$, $\Delta_{kl} f = f(z_{k+1\, l+1}) - f(z_{k+1\, l}) - f(z_{k\, l+1}) + f(z_{kl})$, $\overset{1}{\Delta}_{kl} f = f(z_{k+1\, l}) - f(z_{kl})$ and $\overset{2}{\Delta}_{kl} f = f(z_{k\, l+1}) - f(z_{kl})$.

DEFINITION 3.2.1. Let $X \in L^2(T\times\Omega)$ and $Y \in \mathcal{S}^{2,1}$. We define the following functions, under the condition that the indicated limits exist in $L^2(\Omega)$:

$$(X * Y)_z = \lim_{\|\sigma\|\to 0} \sum_{k,\,l} (\overset{1}{\Delta}_{kl} X)(\overset{2}{\Delta}_{kl} Y) \tag{3.2.2}$$

$$(X \circ Y)_z = \lim_{\|\sigma\|\to 0} \sum_{k,\,l} X_{kl}(\Delta_{kl} Y) \tag{3.2.3}$$

$$[X, Y]_z = \lim_{\|\sigma\|\to 0} \sum_{k,\,l} (\Delta_{kl} X)(\Delta_{kl} Y) \tag{3.2.4}$$

$$\langle X, Y\rangle_{1,z} = \lim_{\|\sigma\|\to 0} \sum_{k,l,l'} (\Delta_{kl} X)(\Delta_{kl'} Y) \tag{3.2.5}$$

$$\langle X, Y\rangle_{2,z} = \lim_{\|\sigma\|\to 0} \sum_{k,k',l} (\Delta_{kl} X)(\Delta_{k'l} Y) \tag{3.2.6}$$

$$\langle\!\!\mid X, Y\rangle_{1,z} = \lim_{\|\sigma\|\to 0} \sum_{k,\,l} (\overset{1}{\Delta}_{kl} X)(\Delta_{kl} Y) \tag{3.2.7}$$

$$\langle\!\!\mid X, Y\rangle_{2,z} = \lim_{\|\sigma\|\to 0} \sum_{k,\,l} (\overset{2}{\Delta}_{kl} X)(\Delta_{kl} Y) \tag{3.2.8}$$

In the sequel we will omit subscript z without confusion.

3.3 Lemmas

LEMMA 3.3.1. *Let* $H \in \mathcal{H}^{4,2}(\Omega)$, $X \in \mathcal{S}^{4,1}$. *Then* $HX_z \in L^2(\Omega)$, *and*

$$\begin{aligned} HX_z = HX_0 &+ \int_{R_z} Hf_\zeta d\zeta + \int_{R_z} H\phi_\zeta dW_\zeta + \int_{R_z} (D_\zeta H)\phi_\zeta d\zeta \\ &+ \int_{R_z \times R_z} H\psi(\zeta, \zeta')dW_\zeta dW_{\zeta'} + \int_{R_z \times R_z} (D_\zeta H)\tilde{\psi}(\zeta, \zeta')d\zeta dW_{\zeta'} \\ &+ \int_{R_z \times R_z} (D_\zeta D_{\zeta'} H)\psi(\zeta, \zeta')d\zeta d\zeta' \\ &+ \int_{R_z \times R_z} Hh(\zeta, \zeta')d\zeta dW_{\zeta'} + \int_{R_z \times R_z} (D_\zeta H)h(\zeta, \zeta')d\zeta d\zeta' \\ &+ \int_{R_z \times R_z} Hg(\zeta, \zeta')d\zeta d\zeta'. \end{aligned} \tag{3.3.1}$$

PROOF: The lemma follows from Proposition 1.2.7.

LEMMA 3.3.2 *Let* $X, Y \in \mathcal{S}^{4,3}$. *Then* $X*Y$ *exists, with*

$$\begin{aligned} X*Y = \int_{R_z \times R_z} I(\zeta \,\bar{\wedge}\, \zeta)\{ & v^x(\zeta\vee\zeta'; \zeta)v^y(\zeta\vee\zeta'; \zeta')d\zeta' d\zeta \\ &+ v^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'} + u^x(\zeta\vee\zeta'; \zeta)v^y(\zeta\vee\zeta'; \zeta')dW_\zeta d\zeta' \\ &+ u^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta'; \zeta')dW_\zeta dW_{\zeta'} + \bar{D}_\zeta v^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta'; \zeta')d\zeta d\zeta' \\ &+ u^x(\zeta\vee\zeta'; \zeta)\bar{D}_\zeta v^y(\zeta\vee\zeta'; \zeta')d\zeta d\zeta' + u^x(\zeta\vee\zeta'; \zeta)\bar{D}_\zeta u^y(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'} \\ &+ \bar{D}_{\zeta'} u^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta'; \zeta')dW_\zeta d\zeta' + \bar{D}_{\zeta'} u^x(\zeta\vee\zeta'; \zeta)\bar{D}_\zeta u^y(\zeta\vee\zeta'; \zeta')d\zeta d\zeta' \} \end{aligned} \tag{3.3.2}$$

Here u^x, v^x, u^y, v^y *are as in* (2.2.3) *and* (2.2.4) *which correspond to* X *and* Y *respectively.*

In particular, if $Y = X$,

$$X*X = \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta)[v(\zeta\vee\zeta'; \zeta)v(\zeta\vee\zeta'; \zeta')d\zeta' d\zeta \; + \; u(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')dW_\zeta dW_{\zeta'}$$

$$+ \; D^-_{\zeta'} u(\zeta\vee\zeta'; \zeta)D^-_\zeta u(\zeta\vee\zeta'; \zeta')d\zeta d\zeta']$$

$$+ \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')[v(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'}$$

$$+ \; u(\zeta\vee\zeta'; \zeta)D^-_\zeta v(\zeta\vee\zeta'; \zeta')d\zeta d\zeta' \; + \; u(\zeta\vee\zeta'; \zeta)D^-_\zeta u(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'}], \qquad (3.3.3)$$

where we omit superscript x *without confusion for* $Y = X$.

PROOF: Put

$$R_{kl} = [0, z_{kl}], \; \Delta^1_{kl} = R_{k+1\,l} \setminus R_{kl} \text{ and } \Delta^2_{kl} = R_{k\,l+1} \setminus R_{kl}.$$

By Proposition 1.2.4,

$$\Delta^2_{kl} Y = \int 1_{\Delta^2_{kl}}(\zeta)[v^y(z_{kl}; \zeta)d\zeta \; + \; u^y(z_{kl}; \zeta)dW_\zeta]$$

$$+ \int 1_{\Delta^2_{kl} \times \Delta^2_{kl}}(\zeta, \zeta')[\psi^y(\zeta, \zeta')dW_\zeta dW_{\zeta'}$$

$$+ \; h^y(\zeta, \zeta')d\zeta dW_{\zeta'} \; + \; g^y(\zeta, \zeta')d\zeta d\zeta'],$$

$$\Delta^1_{kl} X = \int 1_{\Delta^1_{kl}}(\zeta)[v^x(z_{kl}; \zeta)d\zeta + \; u^x(z_{kl}; \zeta)dW_\zeta]$$

$$+ \int 1_{\Delta^1_{kl} \times \Delta^1_{kl}}(\zeta, \zeta')[\psi^x(\zeta, \zeta')dW_\zeta dW_{\zeta'}$$

$$+ \; h^x(\zeta, \zeta')d\zeta dW_{\zeta'} \; + g^x(\zeta, \zeta')d\zeta d\zeta'].$$

The hypotheses imply that

$$v^y(z_{kl}; \cdot), \; v^x(z_{kl}; \cdot) \in \mathcal{H}^{4,2}(T \times \Omega), \qquad (3.3.4)$$

$$u^y(z_{kl}; \cdot), \; u^x(z_{kl}; \cdot) \in \mathcal{H}^{4,3}(T \times \Omega), \qquad (3.3.5)$$

so

$$\int 1_{\Delta^1_{kl}}(\zeta)[v^x(z_{kl};\zeta)d\zeta + u^x(z_{kl};\zeta)dW_\zeta] \in \mathcal{H}^{4,2}(\Omega),$$

and $\Delta^2_{kl} Y \in \mathcal{S}^{4,3}$, $\Delta^1_{kl} X \in \mathcal{H}^{4,2}(\Omega)$. By Lemma 3.3.1, $(\Delta^1_{kl} X)(\Delta^2_{kl} Y) \in L^2(\Omega)$, and

$$(\Delta^1_{kl} X)(\Delta^2_{kl} Y)$$

$$= \int 1_{\Delta^1_{kl}}(\zeta)[v^x(z_{kl};\zeta)d\zeta + u^x(z_{kl};\zeta)dW_\zeta]$$

$$\times \int 1_{\Delta^2_{kl}}(\zeta)[v^y(z_{kl};\zeta)d\zeta + u^y(z_{kl};\zeta)dW_\zeta]$$

$$+ (\Delta^1_{kl} X)\int 1_{\Delta^2_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')[\psi^y(\zeta,\zeta')dW_\zeta dW_{\zeta'} + h^y(\zeta,\zeta')d\zeta dW_{\zeta'} + g^y(\zeta,\zeta')d\zeta d\zeta']$$

$$+ \int 1_{\Delta^1_{kl}\times\Delta^1_{kl}}(\zeta,\zeta')[\psi^x(\zeta,\zeta')dW_\zeta dW_{\zeta'} + h^x(\zeta,\zeta')d\zeta dW_{\zeta'} + g^x(\zeta,\zeta')d\zeta d\zeta']$$

$$\times \int 1_{\Delta^2_{kl}}(\zeta)[v^y(z_{kl};\zeta)d\zeta + u^y(z_{kl};\zeta)dW_\zeta].$$

So

$$\sum_{k,l} (\Delta^1_{kl} X)(\Delta^2_{kl} Y)$$

$$= \sum_{k,l} \int 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')\{v^x(z_{kl};\zeta)v^y(z_{kl};\zeta')d\zeta d\zeta'$$

$$+ v^x(z_{kl};\zeta)u^y(z_{kl};\zeta')d\zeta dW_{\zeta'} + D_{\zeta'} v^x(z_{kl};\zeta)u^y(z_{kl};\zeta')d\zeta d\zeta'$$

$$+ u^x(z_{kl};\zeta)v^y(z_{kl};\zeta')dW_\zeta d\zeta' + u^x(z_{kl};\zeta)D_\zeta v^y(z_{kl};\zeta')d\zeta d\zeta'$$

$$+ u^x(z_{kl};\zeta)u^y(z_{kl};\zeta')dW_\zeta dW_{\zeta'} + u^x(z_{kl};\zeta)D_\zeta u^y(z_{kl};\zeta')d\zeta dW_{\zeta'}$$

$$+ D_{\zeta'} u^x(z_{kl};\zeta)u^y(z_{kl};\zeta')dW_\zeta d\zeta' + D_\zeta u^y(z_{kl};\zeta')D_{\zeta'} u^x(z_{kl};\zeta)d\zeta d\zeta'\}$$

$$+ \sum_{k,\, l} (\Delta^1_{kl} X) \int 1_{\Delta^2_{kl} \times \Delta^2_{kl}}(\zeta, \zeta')[\psi^y(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$

$$+ \; h^y(\zeta, \zeta') d\zeta dW_{\zeta'} \; + \; g^y(\zeta, \zeta') d\zeta d\zeta']$$

$$+ \sum_{k,\, l} \int 1_{\Delta^1_{kl} \times \Delta^1_{kl}}(\zeta, \zeta')[\psi^x(\zeta, \zeta') dW_\zeta dW_{\zeta'} \; + \; h^x(\zeta, \zeta') d\zeta dW_{\zeta'}$$

$$+ \; g^x(\zeta, \zeta') d\zeta d\zeta'] \times \int 1_{\Delta^2_{kl}}(\zeta')[v^y(z_{kl}; \zeta') d\zeta' \; + \; u^y(z_{kl}; \zeta') dW_{\zeta'}]$$

$$= \; I_1 + I_2 + I_3, \quad \text{say.}$$

By the assumption, we can show that as $\|\sigma\| \to 0$,

(a) $I_1 \to$ right hand side of (3.3.2),

(b) $I_2 \to 0$, $I_3 \to 0$,

in $L^2(\Omega)$.

We only check (a). The proof of (b) is analogous. Consider the multiple Skorohod integral in I_1. We want to show

$$\sum_{k,\, l} \int 1_{\Delta^2_{kl} \times \Delta^2_{kl}}(\zeta, \zeta') u^x(z_{kl}; \zeta) u^y(z_{kl}; \zeta') dW_\zeta dW_{\zeta'}$$

$$\to \int_{R_z \times R_z} I(\zeta \overline{\wedge} \zeta) u^x(\zeta \vee \zeta'; \zeta) u^y(\zeta \vee \zeta'; \zeta') dW_\zeta dW_{\zeta'}$$

in $L^2(\Omega)$, as $\|\sigma\| \to 0$.

It suffices to show

$$\sum_{k,\, l} 1_{\Delta^1_{kl} \times \Delta^2_{kl}}(\zeta, \zeta') u^x(z_{kl}; \zeta) u^y(z_{kl}; \zeta') \to I(\zeta \overline{\wedge} \zeta) u^x(\zeta \vee \zeta'; \zeta) u^y(\zeta \vee \zeta'; \zeta')$$

in $\mathcal{H}^{(2)}(T^2\times\Omega)$, as $\|\sigma\| \to 0$. Obviously the convergence holds in $L^2(T^2\times\Omega)$. It remains to show that

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')D_\alpha u^x(z_{kl};\zeta)u^y(z_{kl};\zeta')$$

$$\to I(\zeta\barwedge\zeta)D_\alpha u^x(\zeta\vee\zeta';\zeta)u^y(\zeta\vee\zeta';\zeta') \tag{3.3.6}$$

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')u^x(z_{kl};\zeta)D_\alpha u^y(z_{kl};\zeta')$$

$$\to I(\zeta\barwedge\zeta)u^x(\zeta\vee\zeta';\zeta)D_\alpha u^y(\zeta\vee\zeta';\zeta') \tag{3.3.7}$$

in $L^2(T^3\times\Omega)$,

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')D^2_{\alpha\beta}u^x(z_{kl};\zeta)u^y(z_{kl};\zeta')$$

$$\to I(\zeta\barwedge\zeta)D^2_{\alpha\beta}u^x(\zeta\vee\zeta';\zeta)u^y(\zeta\vee\zeta';\zeta') \tag{3.3.8}$$

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')D_\alpha u^x(z_{kl};\zeta)D_\beta u^y(z_{kl};\zeta')$$

$$\to I(\zeta\barwedge\zeta)D_\alpha u^x(\zeta\vee\zeta';\zeta)D_\beta u^y(\zeta\vee\zeta';\zeta') \tag{3.3.9}$$

and

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')u^x(z_{kl};\zeta)D^2_{\alpha\beta}u^y(z_{kl};\zeta')$$

$$\to I(\zeta\barwedge\zeta)u^x(\zeta\vee\zeta';\zeta)D^2_{\alpha\beta}u^y(\zeta\vee\zeta';\zeta') \tag{3.3.10}$$

in $L^2(T^4\times\Omega)$. Indeed, to prove (3.3.6), we consider

$$\Big|\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')D_\alpha u^x(z_{kl};\zeta)u^y(z_{kl};\zeta') - I(\zeta\barwedge\zeta)D_\alpha u^x(\zeta\vee\zeta';\zeta)u^y(\zeta\vee\zeta';\zeta')\Big|$$

$$\le \Big|\sum_{k,\,l} D_\alpha u^x(\zeta\vee\zeta';\zeta)(u^y(z_{kl};\zeta') - u^y(\zeta\vee\zeta';\zeta'))1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')\Big|$$

$$+\Big|\sum_{k,\,l} u^y(z_{kl};\,\zeta')(D_\alpha u^x(z_{kl};\,\zeta) - D_\alpha u^x(\zeta\vee\zeta';\,\zeta))1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\,\zeta')\Big| \qquad (3.3.11)$$

Now on $\Delta^1_{kl}\times\Delta^2_{kl}$ we have

$$u^y(\zeta\vee\zeta';\zeta') - u^y(z_{kl};\,\zeta') = \int_{R_{\zeta\vee\zeta'}\setminus R_{kl}} [\tilde{\psi}^y(\zeta',\,\zeta'')dW_{\zeta''} + h^y(\zeta'',\,\zeta')d\zeta''].$$

Note $Y\in \mathcal{J}^{4,2}$. By L^p-inequality of Proposition 1.2.3,

$$\begin{aligned}
&E(u^y(\zeta\vee\zeta';\,\zeta') - u^y(z_{kl};\,\zeta'))^4\\
&\quad\le \text{const.}\{E[\int_{R_{\zeta\vee\zeta'}\setminus R_{kl}} (\tilde{\psi}^y(\zeta',\,\zeta''))^2 d\zeta'']^2\\
&\quad + E[\int_{R_{\zeta\vee\zeta'}\setminus R_{kl}} (\int_T (D_\alpha\tilde{\psi}^y(\zeta',\,\zeta''))^2 d\alpha)d\zeta'']^2\\
&\quad + \mu(R_{\zeta\vee\zeta'}\setminus R_{kl})^2 E\int_T (h^y(\zeta',\,\zeta''))^4 d\zeta''\}\\
&\quad\le \text{const. } \mu(R_{\zeta\vee\zeta'}\setminus R_{kl})\{E\int_T (\tilde{\psi}^y(\zeta',\,\zeta''))^4 d\zeta''\\
&\quad + E\int_{T^2} (D_\alpha\tilde{\psi}^y(\zeta',\,\zeta''))^4 d\alpha d\zeta'' + E\int_T (h^y(\zeta',\,\zeta''))^4 d\zeta''\}.
\end{aligned}$$

So

$$\begin{aligned}
&E\int 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\,\zeta')[u^y(\zeta\vee\zeta';\,\zeta') - u^y(z_{kl};\,\zeta')]^4 d\zeta d\zeta'\\
&\qquad\le \text{const. } \|\sigma\|\mu(\Delta^1_{kl}\times\Delta^2_{kl}).
\end{aligned}$$

It follows that the first term of (3.3.11) converges to zero in $L^2(T^3\times\Omega)$. Similarly, the second term converges to zero in $L^2(T^3\times\Omega)$ by the following equation:

$$\begin{aligned}
&D_\alpha u^x(\zeta\vee\zeta';\,\zeta) - D_\alpha u^x(z_{kl};\,\zeta)\\
&\quad = \int_{R_{\zeta\vee\zeta'}\setminus R_{kl}} [D_\alpha\tilde{\psi}^x(\zeta,\,\zeta'')dW_{\zeta''} + D_\alpha h^x(\zeta'',\,\zeta)d\zeta''] + \tilde{\psi}^x(\zeta,\,\alpha)1_{R_{\zeta\vee\zeta'}\setminus R_{kl}}(\alpha).
\end{aligned}$$

Therefore (3.3.6) holds.

The proof of (3.3.7) and (3.3.8) is similar. (3.3.9) and (3.3.10) can be shown by applying the following equation:

$$
\begin{aligned}
&D^2_{\alpha\beta} u^x(\zeta\vee\zeta'; \zeta) - D^2_{\alpha\beta} u^x(z_{kl}; \zeta) \\
&\quad = \int_{R_{\zeta\vee\zeta'}\setminus R_{kl}} (D^2_{\alpha\beta}\tilde{\psi}^x(\zeta, \zeta'')dW_{\zeta''} + D^2_{\alpha\beta} h^x(\zeta'', \zeta)d\zeta'') \\
&\quad + D_\alpha\tilde{\psi}^x(\zeta, \beta)1_{R_{\zeta\vee\zeta'}\setminus R_{kl}}(\beta) + D_\beta\tilde{\psi}^x(\zeta, \alpha)1_{R_{\zeta\vee\zeta'}\setminus R_{kl}}(\alpha).
\end{aligned}
$$

The convergences of other terms in I_1 will follow from:

$$
\begin{aligned}
&\sum_{k,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta, \zeta')\{v^x(z_{kl}; \zeta)v^y(z_{kl}; \zeta') + u^x(z_{kl}; \zeta)D_{\zeta}u^y(z_{kl}; \zeta') \\
&\qquad + D_{\zeta'}v^x(z_{kl}; \zeta)u^y(z_{kl}; \zeta') + D_{\zeta'}u^x(z_{kl}; \zeta)D_{\zeta}u^y(z_{kl}; \zeta')\} \\
&\qquad \to I(\zeta' \barwedge \zeta)\{v^x(\zeta\vee\zeta'; \zeta)v^y(\zeta\vee\zeta'; \zeta') + u^x(\zeta\vee\zeta'; \zeta)\bar{D}_{\zeta}u^y(\zeta\vee\zeta'; \zeta') \\
&\qquad + \bar{D}_{\zeta'}v^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta; \zeta') + \bar{D}_{\zeta}u^x(\zeta\vee\zeta'; \zeta)\bar{D}_{\zeta}u^y(\zeta\vee\zeta'; \zeta')\}
\end{aligned}
$$

in $L^2(T^2\times\Omega)$, as $\|\sigma\| \to 0$,

$$
\begin{aligned}
&\sum_{k,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta, \zeta')\{v^x(z_{kl}; \zeta)u^y(z_{kl}; \zeta') + u^x(z_{kl}; \zeta)D_{\zeta}u^y(z_{kl}; \zeta')\} \\
&\qquad \to I(\zeta' \barwedge \zeta)\{v^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta'; \zeta') + u^x(\zeta\vee\zeta'; \zeta)\bar{D}_{\zeta}u^y(\zeta\vee\zeta'; \zeta')\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta, \zeta')\{u^x(z_{kl}; \zeta)v^y(z_{kl}; \zeta') + D_{\zeta'}u^x(z_{kl}; \zeta)u^y(z_{kl}; \zeta')\} \\
&\qquad \to I(\zeta' \barwedge \zeta)\{u^x(\zeta\vee\zeta'; \zeta)v^y(\zeta\vee\zeta'; \zeta') + \bar{D}_{\zeta'}u^x(\zeta\vee\zeta'; \zeta)u^y(\zeta\vee\zeta'; \zeta')\}
\end{aligned}
$$

in $\mathcal{H}^{(1)}(T^2\times\Omega)$, as $\|\sigma\| \to 0$. We obtain our assertion.

LEMMA 3.3.3. *Suppose that* $X, Y \in \mathcal{S}^{4,3}$. *Then* $[X, Y]$, $\langle\!\!\!-\, X, Y\rangle_i$, $i = 1, 2$ *and* $X \circ Y$ *exist, with*

$$[X, Y] = \int_{R_z} u^x(\zeta; \zeta) u^y(\zeta; \zeta) d\zeta + \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta) \tilde{\psi}^x(\zeta, \zeta') \tilde{\psi}^y(\zeta, \zeta') d\zeta d\zeta' \tag{3.3.12}$$

$$\begin{aligned}\langle\!\!\!-\, X, Y\rangle_1 = \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta) \{ & u^x(\zeta \vee \zeta'; \zeta) \tilde{\psi}^y(\zeta, \zeta') d\zeta dW_{\zeta'} \\ & + u^x(\zeta \vee \zeta'; \zeta) h^y(\zeta', \zeta) d\zeta d\zeta' + D^{-}_{\zeta'} u^x(\zeta \vee \zeta'; \zeta) \tilde{\psi}^y(\zeta, \zeta') d\zeta d\zeta' \}\end{aligned} \tag{3.3.13}$$

$$\begin{aligned}\langle\!\!\!-\, X, Y\rangle_2 = \int_{R_z \times R_z} \mathrm{I}(\zeta \overline{\wedge} \zeta') \{ & u^x(\zeta \vee \zeta'; \zeta) \tilde{\psi}^y(\zeta, \zeta') d\zeta dW_{\zeta'} \\ & + u^x(\zeta \vee \zeta'; \zeta) h^y(\zeta', \zeta) d\zeta d\zeta' + D^{-}_{\zeta} u^x(\zeta \vee \zeta'; \zeta) \tilde{\psi}^y(\zeta, \zeta') d\zeta d\zeta' \}\end{aligned} \tag{3.3.14}$$

$$\begin{aligned}X \circ Y = {} & \int_{R_z} X_\zeta [v^y(\zeta; \zeta) d\zeta + u^y(\zeta; \zeta) dW_\zeta] + \int_{R_z} D^{-}_{\zeta} X_\zeta u^y(\zeta; \zeta) d\zeta \\ & + \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta) X_{\zeta \vee \zeta'} \tilde{\psi}^y(\zeta, \zeta') dW_\zeta dW_{\zeta'} \\ & + \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta') \{ [X_{\zeta \vee \zeta'} h^y(\zeta, \zeta') + D^{-}_{\zeta} X_{\zeta \vee \zeta'} \tilde{\psi}^y(\zeta, \zeta')] d\zeta dW_{\zeta'} \\ & + [X_{\zeta \vee \zeta'} g^y(\zeta, \zeta') + D^{-}_{\zeta'} X_{\zeta \vee \zeta'} h^y(\zeta, \zeta') + \tfrac{1}{2} D^{-}_{\zeta} D^{-}_{\zeta'} X_{\zeta \vee \zeta'} \tilde{\psi}^y(\zeta, \zeta')] d\zeta d\zeta' \}.\end{aligned} \tag{3.3.15}$$

In particular, if $Y = X$, *then*

$$[X, X] = \int_{R_z} u^2(\zeta; \zeta) d\zeta + \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta) \tilde{\psi}^2(\zeta, \zeta') d\zeta d\zeta' \tag{3.3.16}$$

$$\begin{aligned}\langle\!\!\!-\, X, X\rangle_1 + \langle\!\!\!-\, X, X\rangle_2 = \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta') \{ & u(\zeta \vee \zeta'; \zeta) \tilde{\psi}(\zeta, \zeta') d\zeta dW_{\zeta'} \\ & + u(\zeta \vee \zeta'; \zeta) h(\zeta', \zeta) d\zeta d\zeta' + D^{-}_{\zeta} u(\zeta \vee \zeta'; \zeta) \tilde{\psi}(\zeta, \zeta') d\zeta d\zeta' \}\end{aligned} \tag{3.3.17}$$

PROOF: Put $R_{kl} = [0, z_{kl}]$, $\Delta^1_{kl} = R_{k+1\;l} \setminus R_{kl}$, $\Delta^2_{kl} = R_{k\;l+1} \setminus R_{kl}$ and $\Delta_{kl} = (z_{kl}, z_{k+1\;l+1}]$. Then $R_{k+1\;l+1} = R_{kl} + \Delta^1_{kl} + \Delta^2_{kl} + \Delta_{kl}$, $R_{k+1\;l} = R_{kl} + \Delta^1_{kl}$, $R_{k\;l+1} = R_{kl} + \Delta^2_{kl}$.

We have

$$\begin{aligned}
\Delta_{kl}\, Y = {} & \int_{\Delta_{kl}} [v^y(z_{kl};\, \zeta)d\zeta \;+ u^y(z_{kl};\, \zeta)dW_\zeta] \\
& + \int [1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta, \zeta') \;+\; 1_{\Delta^2_{kl}\times\Delta^1_{kl}}(\zeta, \zeta')]\{\psi^y(\zeta, \zeta')dW_\zeta dW_{\zeta'} \\
& + h^y(\zeta, \zeta')d\zeta dW_{\zeta'} \;+\; g^y(\zeta, \zeta')d\zeta d\zeta'\} \\
& + \int_{D_{kl}} \{\psi^y(\zeta, \zeta')dW_\zeta dW_{\zeta'} \;+\; h^y(\zeta, \zeta')d\zeta dW_{\zeta'} \;+\; g^y(\zeta, \zeta')d\zeta d\zeta'\}, \qquad (3.3.18)
\end{aligned}$$

where $D_{kl} = \Delta_{kl} \times \Delta_{kl} \;+ \Delta^1_{kl} \times \Delta_{kl} \;+ \Delta^2_{kl} \times \Delta_{kl} \;+ \Delta_{kl} \times \Delta^1_{kl} \;+ \Delta_{kl} \times \Delta^2_{kl}$.

Using (3.3.18), the lemma can be proved by the same argument as that of Lemma 3.3.2 and therefore we omit it.

LEMMA 3.3.4. *Suppose* $X, Y \in \mathcal{S}^{4,3}$. *Then* $\langle X, Y\rangle_i$, $i = 1, 2$ *exist, with*

$$\langle X, Y\rangle_i = [X, Y] + \langle\!\!\!-X, Y\rangle_i + \langle\!\!\!-Y, X\rangle_i \quad i = 1, 2 \qquad (3.3.19)$$

PROOF: Notice that

$$\begin{aligned}
& \sum_{k,l,l'} (\Delta_{kl}X)(\Delta_{kl'}Y) \\
& = \sum_{k,l} (\Delta_{kl}X)(\Delta_{kl}Y) \;+\; \sum_{k,l<l'} (\Delta_{kl}X)(\Delta_{kl'}Y) \;+\; \sum_{k,l>l'} (\Delta_{kl}X)(\Delta_{kl'}Y) \\
& = \sum_{k,l} (\Delta_{kl}X)(\Delta_{kl}Y) \;+\; \sum_{k,l} (\Delta^1_{kl}X)(\Delta_{kl}Y) \;+\; \sum_{k,l} (\Delta_{kl}X)(\Delta^2_{kl}Y).
\end{aligned}$$

We then have (3.3.19) from Lemma 3.3.3.

In the sequel we write $\langle X\rangle_i = \langle X, X\rangle_i$, $\langle\!\!-X\rangle_{i.} = \langle\!\!-X, X\rangle_i$, $i = 1, 2$, and $[X] = [X, X]$.

LEMMA 3.3.5. *Let* $X \in \mathcal{S}^{4,3}$. *Suppose that* $(\zeta, \zeta') \in T^2$ *and* $\zeta' \barwedge \zeta$. *Then*

$$u^2(\zeta\vee\zeta'; \zeta) = u^2(\zeta; \zeta) + 2\int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)u(\zeta\vee\alpha; \zeta)\tilde{\psi}(\zeta, \alpha)dW_\alpha$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)[\tilde{\psi}^2(\zeta, \alpha) + 2u(\zeta\vee\alpha; \zeta)h(\alpha,\zeta)$$

$$+ D^-_\alpha u(\zeta\vee\alpha; \zeta)\,\tilde{\psi}(\zeta, \alpha)]d\alpha.$$

PROOF: By the hypothesis, $u(\zeta\vee\zeta'; \zeta)$, $u(\zeta; \zeta) \in \mathcal{H}^{4,1}(\Omega)$. Note

$$u(\zeta\vee\zeta'; \zeta) = u(\zeta; \zeta) + \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)\tilde{\psi}(\zeta, \alpha)dW_\alpha + \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)h(\alpha,\zeta)d\alpha.$$

It follows from the Proposition 1.2.7 that

$$u^2(\zeta\vee\zeta'; \zeta)$$

$$= \{\, u(\zeta; \zeta) + \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)\tilde{\psi}(\zeta, \alpha)dW_\alpha + \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)h(\alpha,\zeta)d\alpha \,\}u(\zeta\vee\zeta'; \zeta)$$

$$= u(\zeta; \zeta)u(\zeta\vee\zeta'; \zeta) + \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)u(\zeta\vee\zeta'; \zeta)[\tilde{\psi}(\zeta, \alpha)dW_\alpha + h(\alpha,\zeta)d\alpha]$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha\barwedge\zeta)\tilde{\psi}(\zeta, \alpha)D_\alpha\, u(\zeta\vee\zeta'; \zeta)d\alpha.$$

Since

$$D_\alpha\, u(\zeta\vee\zeta'; \zeta) = D^-_\alpha u(\zeta\vee\alpha; \zeta) + \tilde{\psi}(\zeta, \alpha)1_{R_{\zeta\vee\zeta'}}(\alpha)I(\alpha\barwedge\zeta),$$

we have

$$u^2(\zeta\vee\zeta'; \zeta) = u^2(\zeta; \zeta)$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\{u(\zeta;\, \zeta)[\tilde{\psi}(\zeta,\, \alpha)dW_\alpha + h(\alpha,\zeta)d\alpha] + \tilde{\psi}(\zeta,\, \alpha)D_\alpha u(\zeta;\, \zeta)\}d\alpha$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\{u(\zeta\vee\alpha;\, \zeta)[\tilde{\psi}(\zeta,\, \alpha)dW_\alpha + h(\alpha,\zeta)d\alpha]$$

$$+ D_\alpha^- u(\zeta\vee\alpha;\, \zeta)\tilde{\psi}(\zeta,\, \alpha)d\alpha + \tilde{\psi}^2(\zeta,\, \alpha)d\alpha\}$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\{[u(\zeta\vee\zeta';\, \zeta) - u(\zeta\vee\alpha;\, \zeta)][\tilde{\psi}(\zeta,\, \alpha)dW_\alpha + h(\alpha,\zeta)d\alpha]$$

$$+ [D_\alpha^- u(\zeta\vee\zeta';\, \zeta) - D_\alpha^- u(\zeta\vee\alpha;\, \zeta)]\tilde{\psi}(\zeta,\, \alpha)d\alpha\,\}$$

$$= S_1 + S_2 + S_3 + S_4, \quad \text{say.}$$

Consider S_4. Notice that $\zeta' \,\overline{\wedge}\, \zeta$, using Propositions 1.2.5 and 1.2.7 we have

$$\int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)[u(\zeta\vee\zeta';\, \zeta) - u(\zeta\vee\alpha;\, \zeta)]\tilde{\psi}(\zeta,\, \alpha)dW_\alpha$$

$$= \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\tilde{\psi}(\zeta,\, \alpha)\int_{R_{\zeta\vee\zeta'}\setminus R_{\zeta\vee\alpha}} [\tilde{\psi}(\zeta,\, \beta)dW_\beta + h(\beta,\, \zeta)d\beta]dW_\alpha$$

$$= \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\int_{R_{\zeta\vee\zeta'}\setminus R_{\zeta\vee\alpha}} \tilde{\psi}(\zeta,\, \alpha)\tilde{\psi}(\zeta,\, \beta)dW_\beta dW_\alpha$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\int_{R_{\zeta\vee\zeta'}\setminus R_{\zeta\vee\alpha}} D_\beta\tilde{\psi}(\zeta,\, \alpha)\tilde{\psi}(\zeta,\, \beta)d\beta dW_\alpha$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha \,\overline{\wedge}\, \zeta)\int_{R_{\zeta\vee\zeta'}\setminus R_{\zeta\vee\alpha}} \tilde{\psi}(\zeta,\, \alpha)h(\zeta,\, \beta)d\beta dW_\alpha$$

$$= \int_{R_{\zeta\vee\zeta'}} I(\beta \,\overline{\wedge}\, \zeta)\int_{R_{\zeta\vee\beta}} I(\alpha \,\overline{\wedge}\, \zeta)\{\,\tilde{\psi}(\zeta,\, \alpha)\tilde{\psi}(\zeta,\, \beta)dW_\alpha dW_\beta$$

$$+ \tilde{\psi}(\zeta,\, \alpha)h(\zeta,\, \beta)dW_\alpha d\beta + D_\beta\tilde{\psi}(\zeta,\, \alpha)\tilde{\psi}(\zeta,\, \beta)dW_\alpha d\beta\}$$

$$= \int_{R_{\zeta\vee\zeta'}} I(\beta \,\overline{\wedge}\, \zeta)\tilde{\psi}(\zeta,\, \beta)\Big(\int_{R_{\zeta\vee\beta}} I(\alpha \,\overline{\wedge}\, \zeta)\tilde{\psi}(\zeta,\, \alpha)dW_\alpha\Big)dW_\beta$$

$$- \int_{R_{\zeta\vee\zeta'}} I(\beta \,\overline{\wedge}\, \zeta)\int_{R_{\zeta\vee\beta}} I(\alpha \,\overline{\wedge}\, \zeta)D_\alpha\tilde{\psi}(\zeta,\, \beta)\tilde{\psi}(\zeta,\, \alpha)d\alpha dW_\beta$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)h(\beta, \zeta)\Big(\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)\tilde{\psi}(\zeta, \alpha)dW_{\alpha}\Big)d\beta$$

$$- \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)D_{\alpha}h(\beta, \zeta)\tilde{\psi}(\zeta, \alpha)d\alpha d\beta$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)\tilde{\psi}(\zeta, \beta)D_{\beta}\tilde{\psi}(\zeta, \alpha)dW_{\beta}\,d\alpha.$$

Similarly,

$$\int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)(u(\zeta\vee\zeta'; \zeta) - u(\zeta\vee\alpha; \zeta))h(\alpha,\zeta)d\alpha$$

$$= \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)\tilde{\psi}(\zeta, \beta)\Big(\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)h(\alpha,\zeta)d\alpha\Big)dW_{\beta}$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)h(\beta, \zeta)\Big(\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)h(\alpha,\zeta)d\alpha\Big)d\beta$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)D_{\alpha}h(\beta, \zeta)\tilde{\psi}(\zeta, \alpha)d\alpha d\beta$$

and

$$\int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)[D^{-}_{\alpha}u(\zeta\vee\zeta'; \zeta) - D^{-}_{\alpha}u(\zeta\vee\alpha; \zeta)]\tilde{\psi}(\zeta, \alpha)d\alpha$$

$$= \int_{R_{\zeta\vee\zeta'}} I(\beta\overline{\wedge}\zeta)\int_{R_{\zeta\vee\beta}} I(\alpha\overline{\wedge}\zeta)\{\tilde{\psi}(\zeta, \alpha)[D_{\alpha}\tilde{\psi}(\zeta, \beta)d\alpha dW_{\beta}$$

$$+ D_{\alpha}h(\beta, \zeta)]d\alpha d\beta + D_{\beta}\tilde{\psi}(\alpha, \zeta)D_{\alpha}\tilde{\psi}(\zeta, \beta)d\alpha d\beta\}.$$

Thus

$$S_2 + S_4 = \int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)\tilde{\psi}(\zeta, \alpha)\{u(\zeta; \zeta)$$

$$+ \int_{R_{\zeta\vee\alpha}} I(\beta\overline{\wedge}\zeta)[\tilde{\psi}(\beta, \zeta)dW_{\beta} + h(\beta, \zeta)d\beta]\}dW_{\alpha}$$

$$+ \int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)h(\alpha, \zeta)\{u(\zeta; \zeta) + \int_{R_{\zeta\vee\alpha}} I(\beta\overline{\wedge}\zeta)[\tilde{\psi}(\beta, \zeta)dW_{\beta} + h(\beta, \zeta)d\beta]\}d\alpha$$

$$+\int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)D_\alpha u(\zeta;\zeta)\tilde{\psi}(\zeta,\alpha)d\alpha$$

$$+\int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)\int_{R_{\zeta\vee\alpha}} I(\beta\overline{\wedge}\zeta)\{\tilde{\psi}(\alpha,\zeta)D_\alpha\tilde{\psi}(\beta,\zeta)dW_\alpha d\beta$$

$$+\ \tilde{\psi}(\alpha,\zeta)D_\alpha h(\beta,\zeta)d\alpha d\beta\ + D_\beta\tilde{\psi}(\alpha,\zeta)D_\alpha\tilde{\psi}(\beta,\zeta)d\alpha d\beta\ \}$$

$$=\int_{R_{\zeta\vee\zeta'}} I(\alpha\overline{\wedge}\zeta)\{\tilde{\psi}(\zeta,\alpha)u(\zeta\vee\alpha;\zeta)dW_\alpha\ +\ h(\alpha,\zeta)u(\zeta\vee\alpha;\zeta)d\alpha$$

$$+\ D^{-}_{\alpha}u(\zeta\vee\alpha;\zeta)\tilde{\psi}(\zeta,\alpha)d\alpha\ \}.$$

The proof of the lemma is finished.

3.4 An Itô Formula

LEMMA 3.4.1 *Let* $F \in C_b^2(\mathbb{R})$ *and suppose* $X \in \mathcal{S}^{8,2}\cap\mathcal{S}^{4,3}$. *Then*

$$F(X)\circ X = \int_{R_z}[F(X_\zeta)v(\zeta;\zeta)d\zeta\ +\ F(X_\zeta)u(\zeta;\zeta)dW_\zeta\ +\ F'(X_\zeta)D^{-}_{\zeta}X_\zeta u(\zeta;\zeta)d\zeta]$$

$$+\ \int_{R_z\times R_z} I(\zeta'\overline{\wedge}\zeta)F(X_{\zeta\vee\zeta'})[\tilde{\psi}(\zeta,\zeta')dW_\zeta dW_{\zeta'}\ +\ \tilde{g}(\zeta,\zeta')d\zeta d\zeta']$$

$$+\ \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta,\zeta')\{[F(X_{\zeta\vee\zeta'})h(\zeta,\zeta') + F'(X_{\zeta\vee\zeta'})D^{-}_{\zeta}X_{\zeta\vee\zeta'}\tilde{\psi}(\zeta,\zeta')]d\zeta dW_{\zeta'}$$

$$+\ F'(X_{\zeta\vee\zeta'})[D^{-}_{\zeta'}X_{\zeta\vee\zeta'}h(\zeta,\zeta')\ +\ \frac{1}{2}D^{-}_{\zeta}D^{-}_{\zeta'}X_{\zeta\vee\zeta'}\tilde{\psi}(\zeta,\zeta')]d\zeta d\zeta'$$

$$+\ \frac{1}{2}F''(X_{\zeta\vee\zeta'})D^{-}_{\zeta}X_{\zeta\vee\zeta'}D^{-}_{\zeta'}X_{\zeta\vee\zeta'}\tilde{\psi}(\zeta,\zeta')d\zeta d\zeta'\}. \tag{3.4.1}$$

PROOF: By the hypothesis $X_{kl} \in \mathcal{H}^{4,2}(\Omega)$, so that $F(X_{kl}) \in \mathcal{H}^{4,1}(\Omega)$. From (3.3.18)

$$\sum_{k,l} F(X_{kl})\Delta_{kl}X$$

$$=\ \sum_{k,l}\int_{\Delta_{kl}}\{F(X_{kl})v(z_{kl};\zeta)d\zeta\ + F(X_{kl})u(z_{kl};\zeta)dW_\zeta + D_\zeta F(X_{kl})u(z_{kl};\zeta)d\zeta\}$$

$$+ \sum_{k,l} F(X_{kl}) \int [1_{\Delta_{kl}^1 \times \Delta_{kl}^2}(\zeta, \zeta') + 1_{\Delta_{kl}^2 \times \Delta_{kl}^1}(\zeta, \zeta')][\psi(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$

$$+ h(\zeta, \zeta') d\zeta dW_{\zeta'} + g(\zeta, \zeta') d\zeta d\zeta']$$

$$+ \sum_{k,l} F(X_{kl}) \int_{D_{kl}} \{\psi(\zeta, \zeta') dW_\zeta dW_{\zeta'} + h(\zeta, \zeta') d\zeta dW_{\zeta'} + g(\zeta, \zeta') d\zeta d\zeta'\}$$

$$= \sum_{k,l} \int_{\Delta_{kl}} \{F(X_{kl}) v(z_{kl}; \zeta) d\zeta + F(X_{kl}) u(z_{kl}; \zeta) dW_\zeta + D_\zeta F(X_{kl}) u(z_{kl}; \zeta) d\zeta\}$$

$$+ \sum_{k,l} \int (1_{\Delta_{kl}^1 \times \Delta_{kl}^2}(\zeta, \zeta') + 1_{\Delta_{kl}^2 \times \Delta_{kl}^1}(\zeta, \zeta'))\{F(X_{kl})[\psi(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$

$$+ h(\zeta, \zeta') d\zeta dW_{\zeta'} + g(\zeta, \zeta') d\zeta d\zeta'] + D_\zeta F(X_{kl}) \tilde{\psi}(\zeta, \zeta') d\zeta dW_{\zeta'}$$

$$+ D_\zeta F(X_{kl}) h(\zeta, \zeta') d\zeta d\zeta' + D_\zeta D_{\zeta'} F(X_{kl}) \psi(\zeta, \zeta') d\zeta d\zeta'\}$$

$$+ \sum_{k,l} F(X_{kl}) \int_{D_{kl}} \{\psi(\zeta, \zeta') dW_\zeta dW_{\zeta'} + h(\zeta, \zeta') d\zeta dW_{\zeta'} + g(\zeta, \zeta') d\zeta d\zeta'\}.$$

By Lemma 2.2.1, each process of X, $v(\cdot, \zeta)$, $u(\cdot, \zeta)$ has a continuous version, for any $\zeta \in T$. The lemma can be proved by using a similar technique of Section 3.3. We only check

$$\sum_{k,l} \int 1_{\Delta_{kl}^1 \times \Delta_{kl}^2}(\zeta, \zeta') F(X_{kl}) \psi(\zeta, \zeta') dW_\zeta dW_{\zeta'}$$

$$\to \int_{R_z \times R_z} I(\zeta' \,\overline{\wedge}\, \zeta) F(X_{\zeta \vee \zeta'}) \psi(\zeta, \zeta') dW_\zeta dW_{\zeta'} \tag{3.4.2}$$

in $L^2(\Omega)$.

It suffices to show

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')F(X_{kl})\psi(\zeta,\zeta') \to I(\zeta'\barwedge\zeta)F(X_{\zeta\vee\zeta'})\psi(\zeta,\zeta') \tag{3.4.3}$$

in $\mathcal{H}^{(2)}(T^2\times\Omega)$, i.e. we need to show:

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')F(X_{kl})\psi(\zeta,\zeta') \to I(\zeta'\barwedge\zeta)F(X_{\zeta\vee\zeta'})\psi(\zeta,\zeta') \tag{3.4.4}$$

in $L^2(T^2\times\Omega)$,

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')F'(X_{kl})D_\alpha X_{kl}\psi(\zeta,\zeta')$$

$$\to I(\zeta'\barwedge\zeta)F'(X_{\zeta\vee\zeta'})D_\alpha X_{\zeta\vee\zeta'}\psi(\zeta,\zeta') \tag{3.4.5}$$

in $L^2(T^3\times\Omega)$,

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')F(X_{kl})D_\alpha\psi(\zeta,\zeta') \to I(\zeta'\barwedge\zeta)F(X_{\zeta\vee\zeta'})D_\alpha\psi(\zeta,\zeta') \tag{3.4.6}$$

in $L^2(T^3\times\Omega)$,

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')F''(X_{kl})D_\beta X_{kl}D_\alpha X_{kl}\psi(\zeta,\zeta')$$

$$\to I(\zeta'\barwedge\zeta)F''(X_{\zeta\vee\zeta'})D_\beta X_{\zeta\vee\zeta'}D_\alpha X_{\zeta\vee\zeta'}\psi(\zeta,\zeta') \tag{3.4.7}$$

in $L^2(T^4\times\Omega)$,

$$\sum_{k,\,l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta,\zeta')F'(X_{kl})[D_\beta D_\alpha X_{kl}\psi(\zeta,\zeta') + D_\alpha X_{kl}D_\beta\psi(\zeta,\zeta')]$$

$$\to I(\zeta'\barwedge\zeta)F'(X_{\zeta\vee\zeta'})[D_\beta D_\alpha X_{\zeta\vee\zeta'}\psi(\zeta,\zeta') + D_\alpha X_{\zeta\vee\zeta'}D_\beta\psi(\zeta,\zeta')] \tag{3.4.8}$$

in $L^2(T^4\times\Omega)$,

$$\sum_{k,\, l} 1_{\Delta^1_{kl}\times\Delta^2_{kl}}(\zeta, \zeta')F(X_{kl})D_\alpha D_\beta \psi(\zeta, \zeta')$$

$$\to I(\zeta \bar{\wedge} \zeta)F(X_{\zeta\vee\zeta'})D_\alpha D_\beta \psi(\zeta, \zeta') \tag{3.4.9}$$

in $L^2(T^4\times\Omega)$.

By a similar argument as in Lemma 3.3.2, convergences (3.4.4)–(3.4.9) follow from the facts:

(*i*) $\{X_z,\ z \in T\}$ has a continuous version,

(*ii*) $X \in \mathcal{H}^{8,1}(T\times\Omega)\cap\mathcal{H}^{4,2}(T\times\Omega)$, $DX \in L^8(T^2\times\Omega)$ $D^2X \in L^4(T^3\times\Omega)$,

(*iii*) $\psi \in \mathcal{H}^{8,2}(T^2\times\Omega)$,

(*iv*) $F \in C^2_b(R)$.

THEOPEM 3.4.2 (Itô Formula) *Let* $F \in C^4_b(\mathbb{R})$ *and suppose* $X \in \mathcal{S}^{8,2}\cap\mathcal{S}^{4,3}$. *Then* $F(X) \in \mathcal{S}^{2,1}$ *and*

$$\begin{aligned} F(X) = F(X_0) &+ F'(X)\circ X + F''(X)\circ(X*X) + \tfrac{1}{2}F''(X)\circ(\langle X\rangle_1 + \langle X\rangle_2 - [X]) \\ &+ \tfrac{1}{2}F^{(3)}(X)\circ(\langle X\rangle_1 * X + X*\langle X\rangle_2 + 2[X, X*X]) \\ &+ \tfrac{1}{4}F^{(4)}(X)\circ(\langle X\rangle_1 * \langle X\rangle_2). \end{aligned} \tag{3.4.10}$$

and all terms in (3.3.10) belong to $\mathcal{S}^{2,1}$.

PROOF: The formula (3.4.10) is the same as (2.3.1) but with new notations. In fact, using the technique of Lemma 3.4.1, we have

$$F'(X_z)\circ X_z$$

$$= \int_{R_z} \{ F'(X_\zeta)v(\zeta; \zeta)d\zeta + F'(X_\zeta)u(\zeta; \zeta)dW_\zeta + F''(X_\zeta)D_{\bar{\zeta}}X_\zeta u(\zeta; \zeta)d\zeta \}$$

$$+ \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta) F'(X_{\zeta \vee \zeta'})[\tilde{\psi}(\zeta, \zeta') dW_\zeta dW_{\zeta'} + \tilde{g}(\zeta, \zeta') d\zeta d\zeta']$$

$$+ \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')\{[F'(X_{\zeta \vee \zeta'}) h(\zeta, \zeta') + F''(X_{\zeta \vee \zeta'}) \tilde{\psi}(\zeta, \zeta') D^-_{\zeta} X_{\zeta \vee \zeta'}] d\zeta dW_{\zeta'}$$

$$+ [F''(X_{\zeta \vee \zeta'}) D^-_{\zeta} X_{\zeta \vee \zeta'} h(\zeta, \zeta') + \frac{1}{2} F''(X_{\zeta \vee \zeta'}) D^-_{\zeta'} D^-_{\zeta} X_{\zeta \vee \zeta'} \tilde{\psi}(\zeta, \zeta')$$

$$+ F^{(3)}(X_{\zeta \vee \zeta'}) D^-_{\zeta'} X_{\zeta \vee \zeta} D^-_{\zeta} X_{\zeta \vee \zeta'} \tilde{\psi}(\zeta, \zeta')] d\zeta d\zeta'\} \qquad (3.4.11)$$

Let $Y = X*X$. From Lemma 3.3.2,

$$Y_z = \int_{R_z \times R_z} (\psi^y(\zeta, \zeta') dW_\zeta dW_{\zeta'} + h^y(\zeta, \zeta') d\zeta dW_{\zeta'} + g^y(\zeta, \zeta') d\zeta d\zeta')$$

where

$$\psi^y(\zeta, \zeta') = I(\zeta' \overline{\wedge} \zeta) u(\zeta \vee \zeta'; \zeta) u(\zeta \vee \zeta'; \zeta')$$

$$h^y(\zeta, \zeta') = 1_{\hat{T}^2}(\zeta, \zeta')[v(\zeta \vee \zeta'; \zeta) u(\zeta \vee \zeta'; \zeta') + u(\zeta \vee \zeta'; \zeta) D^-_{\zeta} u(\zeta \vee \zeta'; \zeta')]$$

$$g^y(\zeta, \zeta') = 1_{\hat{T}^2}(\zeta, \zeta')[u(\zeta \vee \zeta'; \zeta) D^-_{\zeta} v(\zeta \vee \zeta'; \zeta')$$

$$+ \frac{1}{2} v(\zeta \vee \zeta'; \zeta) v(\zeta \vee \zeta'; \zeta') + \frac{1}{2} D^-_{\zeta} u(\zeta \vee \zeta'; \zeta') D^-_{\zeta} u(\zeta \vee \zeta'; \zeta)],$$

thus

$$F''(X_z) \circ (X*X)_z$$

$$= \int_{R_z \times R_z} I(\zeta' \overline{\wedge} \zeta) F''(X_{\zeta \vee \zeta'}) u(\zeta \vee \zeta'; \zeta) u(\zeta \vee \zeta'; \zeta') dW_\zeta dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')\{F''(X_{\zeta \vee \zeta'})[v(\zeta \vee \zeta'; \zeta) u(\zeta \vee \zeta'; \zeta') + u(\zeta \vee \zeta'; \zeta) D^-_{\zeta} u(\zeta \vee \zeta'; \zeta')]$$

$$+ F^{(3)}(X_{\zeta \vee \zeta'}) u(\zeta \vee \zeta'; \zeta) u(\zeta \vee \zeta'; \zeta') D^-_{\zeta} X_{\zeta \vee \zeta'}\} d\zeta dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')\{F''(X_{\zeta\vee\zeta'})[u(\zeta\vee\zeta'; \zeta)D^{-}_{\zeta}v(\zeta\vee\zeta'; \zeta')$$

$$+ \frac{1}{2}v(\zeta\vee\zeta'; \zeta)v(\zeta\vee\zeta'; \zeta') + \frac{1}{2}D^{-}_{\zeta}u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta}u(\zeta\vee\zeta'; \zeta)]$$

$$+ F^{(3)}(X_{\zeta\vee\zeta'})[v(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta'}X_{\zeta\vee\zeta}$$

$$+ \frac{1}{2}u(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta'}D^{-}_{\zeta}X_{\zeta\vee\zeta'} + u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta}u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta'}X_{\zeta\vee\zeta'}]$$

$$+ \frac{1}{2}F^{(4)}(X_{\zeta\vee\zeta'})u(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta'}X_{\zeta\vee\zeta}D^{-}_{\zeta}X_{\zeta\vee\zeta'}\}d\zeta d\zeta' \tag{3.4.12}$$

Note

$$\frac{1}{2}(\langle X\rangle_1 + \langle X\rangle_2 - [X])$$

$$= \langle\!\!-X\rangle_1 + \langle\!\!-X\rangle_2 + \frac{1}{2}[X]$$

$$= \frac{1}{2}\int_{R_z} u^2(\zeta; \zeta)d\zeta + \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')\{ \tilde{\psi}(\zeta, \zeta')u(\zeta\vee\zeta'; \zeta)d\zeta dW_{\zeta'}$$

$$+ [u(\zeta\vee\zeta'; \zeta)h(\zeta, \zeta') + \frac{1}{4}\tilde{\psi}^2(\zeta, \zeta') + \tilde{\psi}(\zeta, \zeta')D^{-}_{\zeta}u(\zeta\vee\zeta'; \zeta')]d\zeta d\zeta'\},$$

so

$$\frac{1}{2}F''(X)\circ(\langle X\rangle_1 + \langle X\rangle_2 - [X])$$

$$= \frac{1}{2}\int_{R_z} F''(X_\zeta)u^2(\zeta; \zeta)d\zeta$$

$$+ \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')F''(X_{\zeta\vee\zeta'})\tilde{\psi}(\zeta, \zeta')u(\zeta\vee\zeta'; \zeta)d\zeta dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')\{F''(X_{\zeta\vee\zeta'})[u(\zeta\vee\zeta'; \zeta)h(\zeta, \zeta') + \frac{1}{4}\tilde{\psi}^2(\zeta, \zeta')$$

$$+ \tilde{\psi}(\zeta, \zeta')D^{-}_{\zeta}u(\zeta\vee\zeta'; \zeta')] \; + F^{(3)}(X_{\zeta\vee\zeta'})\tilde{\psi}(\zeta, \zeta')u(\zeta\vee\zeta';\zeta')D^{-}_{\zeta'}X_{\zeta\vee\zeta'}\}d\zeta d\zeta' \tag{3.4.13}$$

Let $Y = \langle X\rangle_1$. By (3.3.19)

$$u^{y}(z; \zeta) = \int_{R_z} I(\zeta\overline{\wedge}\alpha)u(\zeta\vee\alpha; \zeta)\tilde{\psi}(\zeta, \alpha)d\alpha$$

$$v^{y}(z; \zeta) = u^2(z; \zeta) + 2\int_{R_z} I(\zeta\overline{\wedge}\alpha)u(\zeta\vee\alpha; \zeta)\tilde{\psi}(\zeta, \alpha)dW_\alpha$$

$$+ \int_{R_z} (I(\zeta\overline{\wedge}\alpha) + I(\alpha\overline{\wedge}\zeta))[\tilde{\psi}^2(\zeta, \alpha) + 2u(\zeta\vee\alpha; \zeta)h(\alpha, \zeta)$$

$$+2D^{-}_{\alpha}u(\zeta\vee\alpha; \zeta)\tilde{\psi}(\zeta, \alpha)]d\alpha.$$

It follows from Lemma 3.3.5 that

$$v^{y}(\zeta\vee\zeta'; \zeta) = u^2(\zeta\vee\zeta'; \zeta), \quad \text{if} \quad \zeta'\overline{\wedge}\zeta .$$

By (3.3.2)

$$\langle X\rangle_1 * X = \int_{R_z\times R_z} I(\zeta'\overline{\wedge}\zeta)\{u^2(\zeta\vee\zeta'; \zeta)v(\zeta\vee\zeta'; \zeta')d\zeta d\zeta'$$

$$+ u^2(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'}\}$$

$$X * \langle X\rangle_2 = \int_{R_z\times R_z} I(\zeta\overline{\wedge}\zeta')\{u^2(\zeta\vee\zeta'; \zeta)v(\zeta\vee\zeta'; \zeta')d\zeta d\zeta'$$

$$+ u^2(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'}\}$$

$$\langle X\rangle_1 * \langle X\rangle_2 = \int_{R_z\times R_z} I(\zeta'\overline{\wedge}\zeta)\{u^2(\zeta\vee\zeta'; \zeta)u^2(\zeta\vee\zeta'; \zeta')d\zeta d\zeta'.$$

So

$$\frac{1}{2} F^{(3)}(X_z)\circ(\langle X\rangle_1 * X + X * \langle X\rangle_2)_z$$

$$= \frac{1}{2} \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')\{F^{(3)}(X_{\zeta\vee\zeta'})[u^2(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')d\zeta dW_{\zeta'}$$

$$+ \; u^2(\zeta\vee\zeta'; \zeta)v(\zeta\vee\zeta'; \zeta')d\zeta d\zeta']$$

$$+ \; F^{(4)}(X_{\zeta\vee\zeta'})u^2(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')D^{-}_{\zeta}X_{\zeta\vee\zeta'}\}d\zeta d\zeta' \tag{3.4.14}$$

$$\frac{1}{4}F^{(4)}(X_z)\circ(\langle X\rangle_{1,z} * \langle X\rangle_{2,z})$$

$$= \frac{1}{4}\int_{R_z \times R_z} I(\zeta \bar{\wedge} \zeta')u^2(\zeta\vee\zeta'; \zeta)u^2(\zeta\vee\zeta'; \zeta')d\zeta d\zeta'. \tag{3.4.15}$$

Finally, it follows from (3.3.3) and (3.3.12) that

$$[X, X*X] = \frac{1}{2} \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')u(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')\tilde{\psi}(\zeta, \zeta')d\zeta d\zeta',$$

so

$$F^{(3)}(X_z)\circ[X, X*X]_z$$

$$= \frac{1}{2} \int_{R_z \times R_z} 1_{\hat{T}^2}(\zeta, \zeta')F^{(3)}(X_{\zeta\vee\zeta'})u(\zeta\vee\zeta'; \zeta)u(\zeta\vee\zeta'; \zeta')\tilde{\psi}(\zeta, \zeta')d\zeta d\zeta'. \tag{3.4.16}$$

Combining (3.4.11) through (3.4.16), we obtain the Itô formula Theorem 2.3.2 by (3.4.10).

REMARK 3.4.3. Using the localization argument, we may drop the boundness requirement of F, provided we consider all integrals are defined by local procedures.

3.5. The Itô Formula in the Stratonovich Form

Fix $z \in T$ and let $\sigma = \{z_{kl}\}$ be finite partition of $[0, z]$ defined in Section 3.2.

DEFINITION 3.5.1. Let $Y \in \mathcal{S}^{2,1}$. The process $X = \{X_z, z \in T\}$ is said to be *Stratonovich integrable with respect to* Y if there exists a random variable denoted by $X \dot{-} Y$ such that the limit

$$lim_{\|\sigma\| \to 0} \sum_{k,\, l} \frac{1}{4}(X_{k+1\, l+1} + X_{k\, l+1} + X_{k+1\, l} + X_{kl})\Delta_{kl}Y \tag{3.5.1}$$

exists in $L^2(\Omega)$.

LEMMA 3.5.2. *Let* $X, Y \in \mathcal{S}^{4,3}$. *Then* X *is Stratonovich integrable with respect to* Y, *and*

$$X \dot{-} Y = X \circ Y + \frac{1}{4}[X, Y] + \frac{1}{2}\langle X, Y\rangle_1 + \frac{1}{2}\langle X, Y\rangle_2 \tag{3.5.2}$$

PROOF: Write

$$\sum_{k,\, l} \frac{1}{4}(X_{k+1\, l+1} + X_{k\, l+1} + X_{k+1\, l} + X_{kl})\Delta_{kl}Y$$

$$= \sum_{k,\, l} X_{kl}(\Delta_{kl}Y) + \frac{1}{4}\sum_{k,\, l} (\Delta_{kl}X)(\Delta_{kl}Y)$$

$$+ \frac{1}{2}\sum_{k,\, l} (\Delta^1_{kl}X)(\Delta_{kl}Y) + \frac{1}{2}\sum_{k,\, l} (\Delta_{kl}X)(\Delta^2_{kl}Y).$$

The assertion follows from Lemma 3.3.3.

For $X, Y \in \mathcal{S}^{4,3}$, we define the Stratonovich version of $X * Y$ by

$$X \mathbin{\underline{*}} Y = X * Y + \frac{1}{4}[X, Y] + \frac{1}{2}\langle X, Y\rangle_1 + \frac{1}{2}\langle X, Y\rangle_2 \tag{3.5.3}$$

LEMMA 3.5.3. *Let* $X, Y, Z \in \mathcal{S}^{4,3}$. *Then*

$$\langle X, \langle Y, Z\rangle_2\rangle_1 = [X * Y, Z] \tag{3.5.4}$$

$$\langle X, \langle Y, Z\rangle_1\rangle_2 = [Y * X, Z] \tag{3.5.5}$$

$$\langle\!\!\!-\, Y,\ Z * X\rangle_1 = \langle Y, Z\rangle_1 * X \tag{3.5.6}$$

$$\langle\!\!\!-\, Y,\ X * Z\rangle_2 = X * \langle Y, Z\rangle_2 \tag{3.5.7}$$

$$[X, X \overset{*}{-} Y] = [X, X * Y] \tag{3.5.8}$$

Further, let $F \in C_b^2(\mathbb{R})$ *and suppose* $X, Y \in \mathcal{S}^{8,2} \cap \mathcal{S}^{4,3}$. *Then*

$$\langle\!\!\!-\, F(\mathrm{X}), Y\rangle_i = F'(X) \circ \langle\!\!\!-\, X, Y\rangle_i, \quad i = 1, 2 \tag{3.5.9}$$

$$[F(X), Y] = F'(X) \circ [X,\ Y] + F''(X) \circ [X * Y,\ Y]. \tag{3.5.10}$$

PROOF: Equations (3.5.4) through (3.5.10) can by proved by using the definitions of $\langle\!\!\!-\ \rangle_i$, $\dot{-}$, $\overset{*}{-}$, and the representations (3.3.2), (3.3.12) to (3.4.14).

THEOREM 3.5.4. (Itô Formula in Stratonovich Form) *Let* $F \in C_b^4(R)$ *and suppose* $X \in \mathcal{S}^{8,2} \cap \mathcal{S}^{4,3}$. *Then*

$$F(X) - F(X_0) = F'(X) \,\dot{-}\, X + F''(X) \,\dot{-}\, (X \overset{*}{-} X). \tag{3.5.11}$$

and all terms in (3.5.11) *are in* $\mathcal{S}^{2,1}$.

PROOF: Applying (3.5.2) to (3.5.10) for $X = Y = Z$, we have

$$\begin{aligned} F'(X) \,\dot{-}\, X &= F'(X) \circ X + \tfrac{1}{4} F^{(3)}(X) \circ [X, X*X] \\ &\quad + \tfrac{1}{4} F''(X) \circ \{\langle X\rangle_1 + \langle X\rangle_2 - [X]\} \end{aligned} \tag{3.5.12}$$

$$\begin{aligned} F''(X) \,\dot{-}\, (X \overset{*}{-} X) &= F''(X) \circ (X \overset{*}{-} X) + \tfrac{1}{4}[F''(X), X \overset{*}{-} X] \\ &\quad + \tfrac{1}{2} \langle\!\!\!-\, F''(\mathrm{X}), X \overset{*}{-} X\rangle_1 + \tfrac{1}{2} \langle\!\!\!-\, F''(\mathrm{X}), X \overset{*}{-} X\rangle_2 \\ &= F''(X) \circ (X*X) + F''(X) \circ \{\tfrac{1}{4}[X] + \tfrac{1}{2}\langle\!\!\!-\, X\rangle_1 + \tfrac{1}{2}\langle\!\!\!-\, X\rangle_2\} \end{aligned}$$

$$+ \frac{1}{4} F^{(3)}(X)\circ[X, X*X] + \frac{1}{4} F^{(4)}(X)\circ(\langle X\rangle_1 * \langle X\rangle_2)$$

$$+ \frac{1}{2} F^{(3)}(X)\circ(\langle X\rangle_1 * X + X * \langle X\rangle_2) + \frac{1}{2} F^{(3)}(X)\circ[X, X*X]. \qquad (3.5.13)$$

By Theorem 3.4.2, (3.5.12) and (3.5.13) yield (3.5.11).

Again we can remove the boundness requirement of F, provided we define the Stratonovich integral by local procedures.

REMARK As in the adapted case (Hajek [1]), the formula (3.5.11) shows that when the Skorohod -Stratonovich integral is used, the differentiation formula formally agrees with the ordinary rule of calculus.

References

1. B. Hajek, *Stochastic equations of hyperbolic type and a two-parameter Stratonovich calculus.* Ann. Probab. **10** (1982) 451-463
2. D. Nualart and E. Pardoux, *Stochastic calculus with anticipating integrands.* Probab. Th. Rel. Fields **78** (1988) 535- 581

Chapter 4

AN ITÔ-VENTZELL FORMULA FOR TWO-PARAMETER SKOROHOD INTEGRALS

4.1 Introduction

Let $\{X_t, t \geq 0\}$ and $\{F_t, t \geq 0\}$ be respectively an $\mathbb{R}$ and an $L^2(\mathbb{R})$-valued processes satisfying

$$X_t = X_0 + \int_0^t A_s ds + \int_0^t B_s dW_s \tag{4.1.1}$$

$$F_t(x) = F_0(x) + \int_0^t G_s(x) ds + \int_0^t H_s(x) dW_s. \tag{4.1.2}$$

Ventzell [2] gave the differential of the process $F_t(x)$ when (4.1.1) and (4.1.2) are Itô processes, i.e., integrands are adapted to a filtration $\{\mathcal{F}_t, t \geq 0\}$ with respect to which $\{W_t\}$ is a standard Wiener process. Considering generalized stochastic integrals with anticipating integrands, Ocone and Pardoux [1] established a generalized Itô-Ventzell formula for $F_t(X_t)$.

In this chapter we deal with two-parameter anticipative processes. Section 4.2 reviews the definition of the Skorohod integral for Hilbert-space valued processes. Section 4.3 gives an Itô formula for the inner product of $L^2(\mathbb{R})$-valued processes. Using Ocone and Pardoux's technique for the one-parameter case, in Section 4.4, we establish the Itô-Ventzell formula for $F_z(X_z)$, when

$$X_z = X_0 + \int_{R_z} A_\zeta d\zeta + \int_{R_z} B_\zeta dW_\zeta \tag{4.1.3}$$

$$F_z(x) = F_0(x) + \int_{R_z} G_\zeta(x) d\zeta + \int_{R_z} H_\zeta(x) dW_\zeta. \tag{4.1.4}$$

4.2. The Skorohod Integral for Hilbert-Space Valued Processes

In this section we review the definition of the Skorohod integral of a process taking values in a Hilbert space. The definition for the one-parameter case can be found in [1] and it requires no change for higher dimensions.

Let K be a separable real Hilbert space and $(\Omega, \mathcal{F}, P)$ be defined as Chapter 1. By analogy with the one-parameter case [1], we can define D a closed unbounded operator from $L^2(\Omega; K)$ into $L^2(T\times\Omega; K)$. For integers $n \geq 1, k \geq 0$ and $p \in [2, \infty)$, we define $\mathcal{H}^{(n)}(T^k\times\Omega; K)$ and $\mathcal{H}^{p,n}(T^k\times\Omega; K)$ in a way similar to the scalar case. For $u \in \mathcal{H}^{(k)}(T^k\times\Omega; K)$, we can define

$$\delta^k u = \int_{T^k} u_s d^k W_s$$

as the element of $L^2(\Omega; K)$ such that for any $F \in \mathcal{H}^{(k)}(\Omega; K)$,

$$E\langle F, \delta^k u\rangle = E \int_{T^k} \langle D^k F, u_s\rangle ds$$

where $\langle\, , \rangle$ denotes the scalar product in K. It follows from the definition that for any $v \in K$, $u \in \mathcal{H}^{(k)}(T^k\times\Omega; K)$ we have

$$\langle v, \delta^k u\rangle = \int_{T^k} \langle v, u_s\rangle d^k W_s. \tag{4.2.1}$$

4.3. An Itô Formula for the Inner Product

Consider $L^2(\mathbb{R})$-valued processes, where we omit x in $X(x)$ and $Y(x)$:

$$X_z = X_0 + \int_{R_z} f_\zeta d\zeta + \int_{R_z} \phi_\zeta dW_\zeta + \int_{R_z\times R_z} \psi(\zeta, \zeta')dW_\zeta dW_{\zeta'}$$

$$+ \int_{R_z\times R_z} h(\zeta, \zeta')d\zeta dW_{\zeta'} + \int_{R_z\times R_z} g(\zeta, \zeta')d\zeta d\zeta \tag{4.3.1}$$

$$Y_z = Y_0 + \int_{R_z} G_\zeta d\zeta + \int_{R_z} H_\zeta dW_\zeta. \tag{4.3.2}$$

THEOREM 4.3.1. *Let X and Y be $K \equiv L^2(\mathbb{R})$-valued processes given by* (4.3.1) *and* (4.3.2) *respectively. Suppose that*

(*i*) *the processes X_z, Y_z have continuous versions,*

(*ii*) $X_0 \in \mathcal{H}^{4,1}(\Omega; K)$, $f \in \mathcal{H}^{4,1}(T\times\Omega; K)$, $\phi \in \mathcal{H}^{4,2}(T\times\Omega; K)$,

$\psi \in \mathcal{H}^{4,3}(T^2\times\Omega; K)$, $h \in \mathcal{H}^{4,2}(T^2\times\Omega; K)$, $g \in \mathcal{H}^{4,1}(T^2\times\Omega; K)$

(*iii*) $Y_0 \in \mathcal{H}^{4,2}(\Omega; K)$, $G \in \mathcal{H}^{4,2}(T\times\Omega; K)$, $H \in \mathcal{H}^{4,3}(T\times\Omega; K)$,

then

$$\begin{aligned}<X_z, Y_z> = <X_0, Y_0> &+ \int_{R_z} S_1(\zeta)d\zeta + \int_{R_z} S_2(\zeta)dW_\zeta \\ &+ \int_{R_z\times R_z} I(\zeta\,\overline{\wedge}\,\zeta')S_3(\zeta, \zeta')dW_\zeta dW_{\zeta'} + \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta, \zeta')S_4(\zeta, \zeta')d\zeta dW_{\zeta'} \\ &+ \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta, \zeta')S_5(\zeta, \zeta')d\zeta d\zeta' \end{aligned} \tag{4.3.3}$$

with

$$\begin{aligned}S_1(\zeta) = &<X_\zeta, G_\zeta> + <Y_\zeta, v(\zeta; \zeta)> + <u(\zeta; \zeta), H_\zeta> \\ &+ <u(\zeta; \zeta), D_\zeta^- Y_\zeta> + <H_\zeta, D_\zeta^- X_\zeta>\end{aligned} \tag{4.3.4}$$

$$S_2(\zeta) = <X_\zeta, H_\zeta> + <Y_\zeta, u(\zeta; \zeta)> \tag{4.3.5}$$

$$S_3(\zeta, \zeta') = <u(\zeta\vee\zeta'; \zeta), H_{\zeta'}> + <H_\zeta, u(\zeta\vee\zeta'; \zeta')> + <Y_{\zeta\vee\zeta'}, \tilde{\psi}(\zeta, \zeta')> \tag{4.3.6}$$

$$\begin{aligned}S_4(\zeta, \zeta') = &<Y_{\zeta\vee\zeta'}, h(\zeta, \zeta')> + <v(\zeta\vee\zeta'; \zeta), H_{\zeta'}> + <G_\zeta, u(\zeta\vee\zeta'; \zeta')> \\ &+ < \tilde{\psi}(\zeta, \zeta'), H_\zeta> + <u(\zeta\vee\zeta'; \zeta), D_\zeta H_{\zeta'}> + <H_\zeta, D_\zeta^- u(\zeta\vee\zeta'; \zeta')>\end{aligned}$$

$$+ <\tilde{\psi}(\zeta, \zeta'), D_{\zeta}^{-} Y_{\zeta \vee \zeta'}> \tag{4.3.7}$$

$$\begin{aligned} S_5(\zeta, \zeta') = \frac{1}{2} <Y_{\zeta \vee \zeta'}, \ \tilde{g}(\zeta, \zeta')> + <H_{\zeta'}, \ h(\zeta, \zeta')> + <G_{\zeta}, v(\zeta \vee \zeta'; \zeta')> \\ + <u(\zeta \vee \zeta'; \zeta), D_{\zeta} G_{\zeta'}> + <H_{\zeta}, D_{\zeta}^{-} v(\zeta \vee \zeta'; \zeta')> + < \tilde{\psi}(\zeta, \zeta'), D_{\zeta} H_{\zeta'}> \\ + \frac{1}{2} <\tilde{\psi}(\zeta, \zeta'), D_{\zeta}^{-} D_{\zeta'}^{-} Y_{\zeta \vee \zeta'}> + <h(\zeta, \zeta'), D_{\zeta}^{-} Y_{\zeta \vee \zeta'}> \\ + <D_{\zeta} H_{\zeta'}, D_{\zeta}^{-} u(\zeta \vee \zeta'; \zeta)> \end{aligned} \tag{4.3.8}$$

where u and v are determined by (2.2.3) *and* (2.2.4) *corresponding to* X.

PROOF: Let $\{e_k, \ k \in N\}$ be an orthonormal basis of $L^2(\mathbb{R})$. From (4.2.1)

$$\begin{aligned} <X_z, e_k> = <X_0, e_k> + \int_{R_z} <f_{\zeta}, e_k> d\zeta + \int_{R_z} <\phi_{\zeta}, e_k> dW_{\zeta} \\ + \int_{R_z \times R_z} <\psi(\zeta, \zeta'), e_k> dW_{\zeta} dW_{\zeta'} \\ + \int_{R_z \times R_z} <h(\zeta, \zeta'), e_k> d\zeta dW_{\zeta'} + \int_{R_z \times R_z} <g(\zeta, \zeta'), e_k> d\zeta d\zeta' \end{aligned}$$

and

$$<Y_z, e_k> = <Y_0, e_k> + \int_{R_z} <G_{\zeta}, e_k> d\zeta + \int_{R_z} <H_{\zeta}, e_k> dW_{\zeta}.$$

Applying Proposition 2.2.3 to $<X_z, e_k><Y_z, e_k>$, we have

$$\begin{aligned} <X_z, e_k><Y_z, e_k> = <X_0, e_k><Y_0, e_k> \\ + \int_{R_z} \{<Y_{\zeta \otimes z}, e_k><v(\zeta \otimes z; \zeta), e_k> + <X_{\zeta \otimes z}, e_k><G_{\zeta}, e_k>\} d\zeta \\ + \int_{R_z} \{<Y_{\zeta \otimes z}, e_k><u(\zeta \otimes z; \zeta), e_k> + <X_{\zeta \otimes z}, e_k><H_{\zeta}, e_k>\} dW_{\zeta} \end{aligned}$$

$$+ \int_{R_z} \{\langle u(\zeta\otimes z;\zeta), e_k\rangle\langle H_\zeta, e_k\rangle + \langle u(\zeta\otimes z;\zeta), e_k\rangle\langle D_\zeta^- Y_{\zeta\otimes z}, e_k\rangle$$

$$+ \langle H_\zeta, e_k\rangle\langle D_\zeta^- X_{\zeta\otimes z}, e_k\rangle\} d\zeta \tag{4.3.9}$$

By (2.2.2)'

$$\langle X_z, e_k\rangle\langle Y_z, e_k\rangle = \langle X_0, e_k\rangle\langle Y_0, e_k\rangle + \int_{R_z} C_1(\zeta)d\zeta + \int_{R_z} C_2(\zeta)dW_\zeta$$

$$+ \int_{R_z\times R_z} I(\zeta' \barwedge \zeta)C_3(\zeta,\zeta')dW_\zeta dW_{\zeta'} + \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta,\zeta')C_4(\zeta,\zeta')d\zeta dW_{\zeta'}$$

$$+ \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta,\zeta')C_5(\zeta,\zeta')d\zeta d\zeta'$$

where

$$C_1(\zeta) = \langle X_\zeta, e_k\rangle\langle G_\zeta, e_k\rangle + \langle Y_\zeta, e_k\rangle\langle v(\zeta;\zeta), e_k\rangle + \langle u(\zeta;\zeta), e_k\rangle\langle H_\zeta, e_k\rangle$$

$$+ \langle u(\zeta;\zeta), e_k\rangle\langle D_\zeta^- Y_\zeta, e_k\rangle + \langle H_\zeta, e_k\rangle\langle D_\zeta^- X_\zeta, e_k\rangle$$

$$C_2(\zeta) = \langle X_\zeta, e_k\rangle\langle H_\zeta, e_k\rangle + \langle Y_\zeta, e_k\rangle\langle u(\zeta;\zeta), e_k\rangle$$

$$C_3(\zeta,\zeta') = \langle u(\zeta\vee\zeta';\zeta), e_k\rangle\langle H_{\zeta'}, e_k\rangle + \langle H_\zeta, e_k\rangle\langle u(\zeta\vee\zeta';\zeta'), e_k\rangle$$

$$+ \langle Y_{\zeta\vee\zeta'}, e_k\rangle\langle \tilde{\psi}(\zeta,\zeta'), e_k\rangle$$

$$C_4(\zeta,\zeta') = \langle Y_{\zeta\vee\zeta'}, e_k\rangle\langle h(\zeta,\zeta'), e_k\rangle + \langle v(\zeta\vee\zeta';\zeta), e_k\rangle\langle H_{\zeta'}, e_k\rangle$$

$$+ \langle G_\zeta, e_k\rangle\langle u(\zeta\vee\zeta';\zeta'), e_k\rangle + \langle \tilde{\psi}(\zeta,\zeta'), e_k\rangle\langle H_\zeta, e_k\rangle$$

$$+ \langle u(\zeta\vee\zeta';\zeta), e_k\rangle\langle D_\zeta H_{\zeta'}, e_k\rangle + \langle H_\zeta, e_k\rangle\langle D_\zeta^- u(\zeta\vee\zeta';\zeta'), e_k\rangle$$

$$+ \langle \tilde{\psi}(\zeta,\zeta'), e_k\rangle\langle D_\zeta^- Y_{\zeta\vee\zeta'}, e_k\rangle$$

$$C_5(\zeta,\zeta') = \frac{1}{2}\langle Y_{\zeta\vee\zeta'}, e_k\rangle\langle \tilde{g}(\zeta,\zeta'), e_k\rangle + \langle H_{\zeta'}, e_k\rangle\langle h(\zeta,\zeta'), e_k\rangle$$

$$+ \ <G_\zeta, e_k><v(\zeta\vee\zeta'; \zeta'), e_k> \ + <u(\zeta\vee\zeta'; \zeta), e_k><D_\zeta G_{\zeta'}, e_k>$$

$$+ \ <H_\zeta, e_k><D_{\bar{\zeta}} v(\zeta\vee\zeta'; \zeta'), e_k> \ + < \tilde{\psi}(\zeta, \zeta'), e_k><D_\zeta H_{\zeta'}, e_k>$$

$$+ \frac{1}{2}<\tilde{\psi}(\zeta, \zeta'), e_k><D_{\bar{\zeta}} D_{\bar{\zeta}'} Y_{\zeta\vee\zeta'}, e_k> \ + \ <h(\zeta, \zeta'), e_k><D_{\bar{\zeta}'} Y_{\zeta\vee\zeta'}, e_k>$$

$$+ <D_\zeta H_{\zeta'}, e_k><D_{\bar{\zeta}'} u(\zeta\vee\zeta'; \zeta), e_k>.$$

By hypotheses

$$X \in \mathcal{H}^{4,1}(T\times\Omega; K), \ \ u \in \mathcal{H}^{4,2}(T^2\times\Omega; K), \ \ v \in \mathcal{H}^{4,1}(T^2\times\Omega; K), \ \ Y \in \mathcal{H}^{4,2}(T\times\Omega; K).$$

Thus

$$S_1 \in L^2(T\times\Omega), \ S_2 \in \mathcal{H}^{(1)}(T\times\Omega), \ S_3 \in \mathcal{H}^{(3)}(T^2\times\Omega),$$

$$S_4 \in \mathcal{H}^{(1)}(T^2\times\Omega), \ S_5 \in L^2(T^2\times\Omega).$$

The theorem is proved by summing both sides of (4.3.3) from $k = 0$ to n, and then letting n tend to zero.

4.4. A Generalized Itô-Ventzell Formula

In this section we present an Itô-Ventzell formula for $F_z(X_z)$ when

$$X_z = X_0 + \int_{R_z} A_\zeta d\zeta + \int_{R_z} B_\zeta dW_\zeta \tag{4.4.1}$$

$$F_z(x) = F_0(x) + \int_{R_z} G_\zeta(x) d\zeta + \int_{R_z} H_\zeta(x) dW_\zeta. \tag{4.4.2}$$

ASSUMPTION 4.4.1. *Let* X *be of the form* (4.4.1) *and satisfy*

(a1) $X_0 \in \mathcal{H}^{16,2}(\Omega)\cap\mathcal{H}^{8,3}(\Omega)$

(a2) $A \in \mathcal{H}^{16,2}(T\times\Omega)\cap\mathcal{H}^{8,3}(T\times\Omega)$

(*a*3) $B \in \mathcal{H}^{16,3}(T\times\Omega)\cap\mathcal{H}^{8,4}(T\times\Omega)\cap L^{32}(T\times\Omega)$.

ASSUMPTION 4.4.2. *Let* F *be of the form* (4.4.2) *and satisfy*

(*b*1) $G_z \in C^2(\mathbb{R}),\ H_z \in C^3(\mathbb{R}),\ F_z \in C^4(\mathbb{R}),\ (z, \omega)$ *a.e.*

(*b*2) $\{H', F, F''\} \subset \mathcal{H}^{4,2}(T\times\Omega; K),\ \{G, G', H, H'', F', F^{(3)}\} \subset \mathcal{H}^{4,1}(T\times\Omega; K)$,

(*b*3) (α, ζ, ω) *a.e.*, $D_\alpha G_\zeta \in C^1(\mathbb{R}),\ D_\alpha H_\zeta \in C^2(\mathbb{R}),\ D_\alpha F_\zeta \in C^3(\mathbb{R})$,

$\{D_\alpha G_\zeta',\ D_\alpha H_\zeta',\ D_\alpha H_\zeta^{(2)},\ D_\alpha F_\zeta^{(i)},\ i = 1, 2, 3\} \subset C^0(\mathbb{R})$

(*b*4) $(\alpha, \beta, \zeta, \omega)$ *a.e.*, $\{D_\alpha D_\beta H_\zeta',\ D_\alpha D_\beta F_\zeta''\} \subset C^0(\mathbb{R}),\ D_\alpha D_\beta F_\zeta \in C^2(\mathbb{R})$

(*b*5) $(DH)' \in \mathcal{H}^{4,1}(T^2\times\Omega; K),\ (DF)'' \in \mathcal{H}^{4,2}(T^2\times\Omega; K)$,

(*b*6) $(\alpha, \beta, \zeta, \omega)$ *a.e.*, $\{D_\alpha([D_\beta H_\zeta]'),\ D_\alpha[D_\beta F_\zeta^{(2)}]\} \subset C^0(\mathbb{R})$,

(*b*7) *For any compact subset* K *of* $\mathbb{R}$, *the following holds*

$$E\int_{T^k} \sup_{x\in K} |\Phi_z(x)|^4 dz < \infty, \quad \Phi \in \mathcal{M}$$

where

$\mathcal{M} = \{G^{1/4},\ G', G'', DG',\ (DG)',\ |DH|^{1/2},\ H^{(i)},\ i = 1, 2, 3,\ DH^{(j)}, j = 1, 2,\ [DH]^{(k)}, k$
$= 1, 2,\ D^2(H'),\ D([DH]'),\ F^{(l)}, l = 1, 2, 3, 4,\ |[DF]'|^{1/2},\ [DF]^{(m)}, m = 2, 3,$
$DF^{(n)}, n = 1, 2, 3,\ [D^2F]^{(2)},\ [DF^{(2)}]',\ D([DF]^{(2)})\ \}$

THEOREM 4.4.3. *Let* (X_z) *and* (F_z) *be respectively an* $\mathbb{R}$ *and an* $L^2(\mathbb{R})$*-valued processes satisfying* (4.4.1) *and* (4.4.2). *We suppose that the Assumptions* 4.4.1 *and* 4.4.2 *are in force. Then we have*

$$F(X_z) = F(X_0) + \int_{R_z} S_1(\zeta)d\zeta + \int_{R_z} S_2(\zeta)dW_\zeta$$
$$+ \int_{R_z\times R_z} I(\zeta \overline{\wedge} \zeta)S_3(\zeta, \zeta')dW_\zeta dW_{\zeta'} + \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta, \zeta')S_4(\zeta, \zeta')d\zeta dW_{\zeta'}$$
$$+ \int_{R_z\times R_z} 1_{\hat{T}^2}(\zeta, \zeta')S_5(\zeta, \zeta')d\zeta d\zeta' \tag{4.4.3}$$

where

$$S_1(\zeta) = G_\zeta(X_\zeta) + F'_\zeta(X_\zeta)A_\zeta + F''_\zeta(X_\zeta)[\tfrac{1}{2} B_\zeta^2 + B_\zeta D_\zeta^- X_\zeta]$$
$$+ [D_\zeta^- F_\zeta]'(X_\zeta)B_\zeta + H'_\zeta(X_\zeta)(B_\zeta + D_\zeta^- X_\zeta)$$

$$S_2(\zeta) = F'_\zeta(X_\zeta)B_\zeta + H_\zeta(X_\zeta)$$

$$S_3(\zeta, \zeta') = F''_{\zeta\vee\zeta'}(X_{\zeta\vee\zeta'})B_\zeta B_{\zeta'} + H'_{\zeta'}(X_{\zeta\vee\zeta'})B_\zeta + H'_\zeta(X_{\zeta\vee\zeta'})B_{\zeta'}$$

$$S_4(\zeta, \zeta') = F''_{\zeta\vee\zeta'}(X_{\zeta\vee\zeta'})[A_\zeta B_{\zeta'} + B_\zeta D_\zeta B_{\zeta'}]$$
$$+ F^{(3)}_{\zeta\vee\zeta'}(X_{\zeta\vee\zeta'})[\tfrac{1}{2} B_\zeta^2 B_{\zeta'} + B_\zeta B_{\zeta'} D_\zeta^- X_{\zeta\vee\zeta'}]$$
$$+ [D_\zeta^- F_{\zeta\vee\zeta'}]^{(2)}(X_{\zeta\vee\zeta'})B_\zeta B_{\zeta'} + H'_{\zeta'}(X_{\zeta\vee\zeta'})[A_\zeta + D_\zeta B_{\zeta'}]$$
$$+ H_\zeta^{(2)}(X_{\zeta\vee\zeta'})[\tfrac{1}{2} B_\zeta^2 + B_\zeta D_\zeta^- X_{\zeta\vee\zeta'}] + H_\zeta^{(2)}(X_{\zeta\vee\zeta'})[B_\zeta B_{\zeta'}$$
$$+ B_{\zeta'} D_\zeta^- X_{\zeta\vee\zeta'}] + [D_\zeta H_{\zeta'}]'(X_{\zeta\vee\zeta'})B_\zeta + G'_{\zeta'}(X_{\zeta\vee\zeta'})B_{\zeta'}$$

$$S_5(\zeta, \zeta') = F''_{\zeta\vee\zeta'}(X_{\zeta\vee\zeta'})[\tfrac{1}{2}A_\zeta A_{\zeta'} + B_\zeta D_\zeta A_{\zeta'} + \tfrac{1}{2} D_\zeta B_{\zeta'} D_{\zeta'} B_\zeta]$$
$$+ F^{(3)}_{\zeta\vee\zeta'}(X_{\zeta\vee\zeta'})[\tfrac{1}{2} B_\zeta^2 A_{\zeta'} + A_\zeta B_{\zeta'} D_\zeta^- X_{\zeta\vee\zeta}$$

$$+ B_\zeta D_\zeta B_{\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'} + B_\zeta B_{\zeta'} D_{\zeta'} B_\zeta + \frac{1}{2} B_\zeta B_{\zeta'} D^-_{\zeta} D^-_{\zeta'} X_{\zeta\vee\zeta'}]$$

$$+ F^{(4)}_{\zeta\vee\zeta'}(X_{\zeta\vee\zeta'})[\frac{1}{8} B^2_\zeta B^2_{\zeta'} + \frac{1}{2} B^2_\zeta B_{\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'}$$

$$+ \frac{1}{2} B_\zeta B_{\zeta'} D^-_\zeta X_{\zeta\vee\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'}]$$

$$+ [D^-_\zeta F(X_{\zeta\vee\zeta'})]^{(2)}[A_{\zeta'} B_\zeta + B_{\zeta'} D_{\zeta'} B_\zeta]$$

$$+ [D^-_\zeta F(X_{\zeta\vee\zeta'})]^{(3)}[\frac{1}{2} B_\zeta B^2_{\zeta'} + B_\zeta B_{\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'}]$$

$$+ \frac{1}{2} [D^-_\zeta D^-_{\zeta'} F_{\zeta\vee\zeta'}]^{(2)}(X_{\zeta\vee\zeta'}) B_\zeta B_{\zeta'} + H'_\zeta(X_{\zeta\vee\zeta'}) D_\zeta A_{\zeta'}$$

$$+ H''_\zeta(X_{\zeta\vee\zeta'})[A_{\zeta'} B_\zeta + B_{\zeta'} D_\zeta B_{\zeta'} + A_{\zeta'} D^-_\zeta X_{\zeta\vee\zeta} + B_{\zeta'} D_\zeta B_{\zeta'}$$

$$+ D_\zeta B_{\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'} + B_{\zeta'} D^-_\zeta D^-_{\zeta'} X_{\zeta\vee\zeta'}]$$

$$+ H^{(3)}_\zeta (X_{\zeta\vee\zeta'})[\frac{1}{2} B_\zeta B^2_{\zeta'} + B_\zeta B_{\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta} + \frac{1}{2} B^2_\zeta D^-_\zeta X_{\zeta\vee\zeta'}$$

$$+ B_{\zeta'} D^-_\zeta X_{\zeta\vee\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'}] + [D_\zeta H_{\zeta'}]'(X_{\zeta\vee\zeta'}) D_{\zeta'} B_\zeta$$

$$+ [D_\zeta H_{\zeta'}]^{(2)}(X_{\zeta\vee\zeta'})[B_\zeta B_{\zeta'} + B_\zeta D^-_{\zeta'} X_{\zeta\vee\zeta'}]$$

$$+ G'_\zeta(X_{\zeta\vee\zeta'}) A_{\zeta'} + G''_\zeta(X_{\zeta\vee\zeta'})[\frac{1}{2} B^2_\zeta + B_{\zeta'} D^-_{\zeta'} X_{\zeta\vee\zeta'}]$$

$$+ [D_\zeta G_{\zeta'}]'(X_{\zeta\vee\zeta'}) B_\zeta.$$

PROOF: We will use Ocone and Pardoux's method [1] to prove the result.

Let $\varphi \in C^\infty_c(\mathbb{R}, \mathbb{R}_+)$ such that $\int_R \varphi(x)dx = 1$. For $\varepsilon > 0$, define

$$\varphi_\varepsilon(x)=\varepsilon^{-1}\varphi(\frac{x}{\varepsilon}).$$

It follows from Corollary 2.3.4 that

$$\begin{aligned}\varphi_\varepsilon(X_z - x) = \varphi_\varepsilon(X_z - x) &+ \int_{R_z} f_x(\zeta)d\zeta + \int_{R_z} \phi_x(\zeta)dW_\zeta \\ &+ \int_{R_z\times R_z} \psi_x(\zeta, \zeta')dW_\zeta dW_{\zeta'} + \int_{R_z\times R_z} h_x(\zeta, \zeta')d\zeta dW_{\zeta'} \\ &+ \int_{R_z\times R_z} g_x(\zeta, \zeta')d\zeta d\zeta', \end{aligned} \tag{4.4.4}$$

where integrands in (4.4.4) are given by (2.3.12)–(2.3.16) with $F = \varphi_\varepsilon(\cdot - x)$, $f = A$, $\phi = B$, and $d = 1$. We can and will regard (4.4.4) as an equality between processes with values in $L^2(\mathbb{R})$, using the same argument as in the proof of Theorem 3.2 [1].

It holds from Theorem 4.3.1

$$\begin{aligned}&\int_R F_z(x)\varphi_\varepsilon(X_z - x)dx \\ &= \int_R F_0(x)\varphi_\varepsilon(X_0 - x)dx + \int_{R_z}\int_R C_1(x; \zeta)dxd\zeta + \int_{R_z}\int_R C_2(x; \zeta)dxdW_\zeta \\ &+ \int_{R_z\times R_z}\int_R I(\zeta'\,\overline{\wedge}\,\zeta)C_3(x; \zeta, \zeta')dxdW_\zeta dW_{\zeta'} \\ &+ \int_{R_z\times R_z}\int_R 1_{\hat{T}^2}(\zeta, \zeta')C_4(x; \zeta, \zeta')dxd\zeta dW_{\zeta'} \\ &+ \int_{R_z\times R_z}\int_R 1_{\hat{T}^2}(\zeta, \zeta')C_5(x; \zeta, \zeta')dxd\zeta d\zeta' \end{aligned} \tag{4.4.5}$$

where

$$\begin{aligned}C_1(x; \zeta) = G_\zeta(x)\varphi_\varepsilon(X_\zeta - x) &+ F_\zeta(x)v_x(\zeta; \zeta) + H_\zeta(x)u_x(\zeta; \zeta) \\ &+ u_x(\zeta; \zeta)D_\zeta^- F_\zeta(x) + H_\zeta(x)D_\zeta^-\varphi_\varepsilon(X_\zeta - x)\end{aligned}$$

$$C_2(x; \zeta) = H_\zeta(x)\varphi_\varepsilon(X_\zeta - x) + F_\zeta(x)u_x(\zeta; \zeta)$$

$$C_3(x;\zeta,\zeta') = H_\zeta(x)u_x(\zeta\vee\zeta';\zeta') + H_{\zeta'}(x)u_x(\zeta\vee\zeta';\zeta) + F_{\zeta\vee\zeta'}(x)\,\widetilde{\psi}_x(\zeta,\zeta')$$

$$C_4(x;\zeta,\zeta') = F_{\zeta\vee\zeta'}(x)h_x(\zeta,\zeta') + H_{\zeta'}(x)v_x(\zeta\vee\zeta';\zeta) + G_\zeta(x)u_x(\zeta\vee\zeta';\zeta')$$

$$+ \widetilde{\psi}_x(\zeta,\zeta')H_\zeta(x) + u_x(\zeta\vee\zeta';\zeta)D_\zeta H_{\zeta'}(x)$$

$$+ H_\zeta(x)D^-_\zeta u_x(\zeta\vee\zeta';\zeta') + \widetilde{\psi}_x(\zeta,\zeta')D^-_\zeta F_{\zeta\vee\zeta'}(x)$$

$$C_5(x;\zeta,\zeta') = \frac{1}{2}F_{\zeta\vee\zeta'}(x)\,\widetilde{g}_x(\zeta,\zeta') + H_{\zeta'}(x)h_x(\zeta,\zeta') + G_\zeta(x)v_x(\zeta\vee\zeta';\zeta')$$

$$+ D_\zeta G_{\zeta'}(x)u_x(\zeta\vee\zeta';\zeta) + H_\zeta(x)D^-_\zeta v_x(\zeta\vee\zeta';\zeta') + \widetilde{\psi}_x(\zeta,\zeta')D_\zeta H_{\zeta'}(x)$$

$$+ \frac{1}{2}\widetilde{\psi}_x(\zeta,\zeta')D^-_\zeta D^-_\zeta F_{\zeta\vee\zeta'}(x) + D_\zeta H_{\zeta'}(x)D^-_\zeta u_x(\zeta\vee\zeta';\zeta)$$

$$+ h_x(\zeta,\zeta')D^-_\zeta F_{\zeta\vee\zeta'}(x),$$

where v_x, u_x are v, u processes corresponding to (4.4.4).

It follows from the same arguments as in the proofs of (2.3.7)–(2.3.10) that for any $x \in \mathbb{R}$, we have

$$u_x(\zeta\vee\zeta';\zeta) = \varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta \tag{4.4.6}$$

$$v_x(\zeta\vee\zeta';\zeta) = \varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)A_\zeta + \frac{1}{2}\varphi''_\varepsilon(X_{\zeta\vee\zeta'} - x)B^2_\zeta$$

$$+ \varphi''_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta D^-_\zeta X_{\zeta\vee\zeta'} \tag{4.4.7}$$

on $\hat{T}^2(\zeta,\zeta')$.

Replace u_x, v_x by (4.4.6) and (4.4.7) in (4.4.5) and then integrate by parts all integrals where derivatives of φ appear. Finally let ε tend to zero. We only check the convergence in the multiple Skorohod integral part. The convergences of other parts can be proved by a similar argument.

We will show that

$$\int_R I(\zeta \,\overline{\wedge}\, \zeta) F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta B_{\zeta'}dx$$

$$\to I(\zeta \,\overline{\wedge}\, \zeta) F''_{\zeta\vee\zeta'}(x)B_\zeta B_{\zeta'} \tag{4.4.8}$$

in $\mathcal{H}^{(2)}(T^2\times\Omega)$.

From the proof of Theorem 3.1 [1] we suppose below that the condition ($b7$) holds with $\sup_{x\in K}$ replaced by $\sup_{x\in R}$. Note

$$\left| \int_R F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta B_{\zeta'}dx \right| \le \sup_{x\in R} |F''_{\zeta\vee\zeta'}(x)|\,|B_\zeta B_{\zeta'}|.$$

It follows from ($b7$) and the continuity of $F''_{\zeta\vee\zeta'}(x)$ at x for fixed (ω, ζ) that the convergence (4.4.8) holds in $L^2(T^2\times\Omega)$.

Moreover,

$$\left| \int_R D_\alpha[F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta B_{\zeta'}]dx \right|$$

$$= \left| \int_R \{D_\alpha F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta B_{\zeta'} + F''_{\zeta\vee\zeta'}(x)\varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\alpha X_{\zeta\vee\zeta'}B_\zeta B_{\zeta'}\right.$$

$$\left. + F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\alpha(B_\zeta B_{\zeta'})\}dx \right|$$

$$\le \sup_{x\in R} \{|D_\alpha F''_{\zeta\vee\zeta'}(x)|\,|B_\zeta B_{\zeta'}| + |F^{(3)}_{\zeta\vee\zeta'}(x)|\,|D_\alpha X_{\zeta\vee\zeta'}|\,|B_\zeta B_{\zeta'}|$$

$$+ |F''_{\zeta\vee\zeta'}(x)|\,|D_\alpha B_\zeta B_{\zeta'} + B_\zeta D_\alpha B_{\zeta'}|\},$$

$$\left| \int_R D_\alpha D_\beta[F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta B_{\zeta'}]dx \right|$$

$$= \left| \int_R \{D_\alpha D_\beta F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)B_\zeta B_{\zeta'}\right.$$

$$+ D_\alpha F''_{\zeta\vee\zeta'}(x)\varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\beta X_{\zeta\vee\zeta'}B_\zeta B_{\zeta'} + D_\alpha F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\beta(B_\zeta B_{\zeta'})$$

$$+ D_\beta F''_{\zeta\vee\zeta'}(x)\varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\alpha X_{\zeta\vee\zeta'}B_\zeta B_{\zeta'}$$

$$+ F''_{\zeta\vee\zeta'}(x)\varphi''_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\alpha X_{\zeta\vee\zeta'}D_\beta X_{\zeta\vee\zeta'}B_\zeta B_{\zeta'}$$

$$+ F''_{\zeta\vee\zeta'}(x)\varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)[D_\alpha D_\beta X_{\zeta\vee\zeta'}B_\zeta B_{\zeta'} + D_\alpha X_{\zeta\vee\zeta'}D_\beta(B_\zeta B_{\zeta'})]$$

$$+ D_\beta F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\alpha(B_\zeta B_{\zeta'}) + F''_{\zeta\vee\zeta'}(x)\varphi'_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\beta X_{\zeta\vee\zeta'}D_\alpha(B_\zeta B_{\zeta'})$$

$$+ F''_{\zeta\vee\zeta'}(x)\varphi_\varepsilon(X_{\zeta\vee\zeta'} - x)D_\alpha D_\beta(B_\zeta B_{\zeta'})\}dx|$$

$$\leq \sup_{x\in R}\{|D_\alpha D_\beta F''_{\zeta\vee\zeta'}(x)||B_\zeta B_{\zeta'}| + |[D_\alpha F''_{\zeta\vee\zeta'}(x)]'||D_\beta X_{\zeta\vee\zeta'}||B_\zeta B_{\zeta'}|$$

$$+ |[D_\alpha F''_{\zeta\vee\zeta'}(x)]||D_\beta(B_\zeta B_{\zeta'})| + |[D_\beta F''_{\zeta\vee\zeta'}(x)]'||D_\alpha X_{\zeta\vee\zeta'}||B_\zeta B_{\zeta'}|$$

$$+ |F^{(4)}_{\zeta\vee\zeta'}(x)||D_\alpha X_{\zeta\vee\zeta'}||D_\beta X_{\zeta\vee\zeta'}||B_\zeta B_{\zeta'}| + |F^{(3)}_{\zeta\vee\zeta'}(x)||D_\alpha D_\beta X_{\zeta\vee\zeta'}||B_\zeta B_{\zeta'}|$$

$$+ |F^{(3)}_{\zeta\vee\zeta'}(x)||D_\alpha X_{\zeta\vee\zeta'}||D_\beta(B_\zeta B_{\zeta'})| + |[D_\beta F''_{\zeta\vee\zeta'}(x)]||D_\alpha(B_\zeta B_{\zeta'})|$$

$$+ |F^{(3)}_{\zeta\vee\zeta'}(x)||D_\beta X_{\zeta\vee\zeta'}||D_\alpha B_\zeta B_{\zeta'}| + |F''_{\zeta\vee\zeta'}(x)||D_\alpha D_\beta(B_\zeta B_{\zeta'})|\}.$$

The convergence (4.4.8) in $\mathcal{H}^{(2)}(T^2\times\Omega)$ then follows from Lebesgue's dominated convergence theorem and the hypotheses $(a1)$ to $(a3)$ and $(b7)$.

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Chapter 5

THE SKOROHOD INTEGRAL REPRESENTATION FOR TWO-PARAMETER PROCESSES

5.1. Introduction

Let $u = \{u_z, z \in T\}$, $T = [0, 1]^d$ be a Skorohod integrable process. The indefinite integral $M_z = \delta(u 1_{[0, z]})$, supposing that it exists for all $z \in T$, has the following property [3]

$$E\Big(\int_{(z, z']} u_\zeta dW_\zeta \mid \mathcal{F}_{(z, z']^c}\Big) = 0, \qquad z < z'. \tag{5.1.1}$$

For $d = 1$, this says:

$$E(M_t - M_s \mid \mathcal{F}_{(s, t]^c}) = 0 \qquad \text{for any } s < t, \tag{5.1.2}$$

and $M_0 = 0$. Duc and Nualart [2] showed that if M has a finite chaos decomposition, then M can be expressed as an indefinite Skorohod integral if and only if

$$V^*(M) \equiv \limsup_{|\Pi| \to 0} E\Big[\sum_{j=0}^{m} (M_{t_{j+1}} - M_{t_j})^2\Big] < \infty, \tag{5.1.3}$$

where $\Pi = \{0 = \tau_0 \le \tau_1 < \cdots < \tau_m \le \tau_{m+1} = 1\}$.

The aim of this chapter is to obtain an analogous result in the two parameter case. The proof of the result for the one parameter case starts from the fact that a multiple Wiener integral can be expressed as an iterated stochastic integral.

In Section 5.2, we study the properties of two-parameter square integrable processes which satisfy

$$E(M(z', z''] \mid \mathcal{F}_{(z', z'']^c}) = 0 \qquad \text{for any } z' < z'', \tag{5.1.4}$$

and

$$M_{s0} = M_{0t} = 0, \qquad 0 \le s, t \le 1 \tag{5.1.5}$$

where $M(z', z''] = M(z'') - M(s', t'') - M(s'', t') + M(z')$, if $z' = (s', t') < z'' = (s'', t'')$.

Section 5.3 obtains the result: suppose that M is a square integrable process with finite chaos decomposition and satisfies (5.1.4) and (5.1.5), then M has a Skorohod integral representation if and only if

$$V^*(M) \equiv limsup_{|\Pi| \to 0} E[\sum_{p,q} M(z_{pq}, z_{p+1\ q+1}]^2] < \infty,$$

where

$$\Pi = \{0 = s_0 \le s_1 < \cdots < s_m \le s_{m+1} = 1,\ 0 = t_0 \le t_1 < \cdots < t_{m'} \le t_{m'+1} = 1\},$$

$m, m' \ge 1$, $|\Pi| = max_{p,q}\{(s_{p+1} - s_p), (t_{q+1} - t_q)\}$, and $z_{pq} = (s_p, t_q)$.

5.2. The Skorohod Representation: Henceforth $d = 2$.

In the sequel our definitions are similar to those in [2].

Denote by $\mathcal{S}$ the class of stochastic processes $M = \{M(z), z \in T\}$ verifying the following properties:

(A1) $E(M_z^2) < +\infty$, for any $z \in T$,

(A2) $M_{s0} = M_{0t} = 0$, $\quad 0 \le s, t \le 1$,

(A3) $E(M(z', z''] \mid \mathcal{F}_{(z', z'']^c}) = 0$ for any $z' < z''$.

DEFINITION 5.2.1. A stochastic process $M = \{M_z, z \in T\}$ is called to possess a *Skorohod Integral Representation* (*SIR*) if there exists a process $u = \{u_z, z \in T\} \in L^2(T\times\Omega)$ such that $u1_{R_z}$ is Skorohod integrable for all $z \in T$ and $M_z = \delta(u1_{R_z})$.

DEFINITION 5.2.2. A stochastic process $M = \{M_z, z \in T\}$ is called to possess a *Skorohod Representation* if there exists a sequence of measurable functions $g_n(z \mid z_1, \dots, z_n)$, $n \geq 0$, defined on the set $T \times T_n$ such that for any $z \in T$, we have

$$\sum_{n=0}^{\infty} \int_{T_{n+1}} \Big(\sum_{i=1}^{n+1} g_n(z_i \mid z_1, \dots, \hat{z}_i, \dots, z_{n+1}) 1_{R_z}(z_i)\Big)^2 \, dz_1 \cdots dz_{n+1} < \infty,$$

and

$$M_z = \sum_{n=0}^{\infty} \int_{T_{n+1}} \Big(\sum_{i=1}^{n+1} g_n(z_i \mid z_1, \dots, \hat{z}_i, \dots, z_{n+1}) 1_{R_z}(z_i)\Big) dW(z_1) \cdots dW(z_{n+1}),$$

where

$$T_n = \{(z_1, \dots, z_n): \; z_i = (s_i, t_i) \in T, i = 1, \dots, n, \; 0 \leq s_1 < \cdots < s_n \leq 1\},$$

and

$$g_n(z_i \mid z_1, \dots, \hat{z}_i, \dots, z_{n+1}) = g_n(z_i \mid z_1, \dots, z_{i-1}, z_{i+1} \dots, z_{n+1}).$$

It follows from Proposition 1.3.1 that the class $\mathcal{S}$ contains all the processes M possessing an *SIR*. Furthermore, considering the converse problem, we have the following result.

THEOREM 5.2.3. *A stochastic process* $M = \{M_z, z \in T\}$ *belongs to the class* $\mathcal{S}$ *if and only if it possesses a Skorohod Representation, and in this case the representation is unique.*

PROOF: First we remark that it follows from the properties of multiple Itô integral ([1]) that for any $f \in L^2(T^n)$, we have

$$I_n(f) = n! I_n(\tilde{f} 1_{T_n}), \tag{5.2.1}$$

where $\tilde{f}$ is the symmetrization of f over T^n.

Suppose $M \in \mathcal{S}$. By (5.2.1), we suppose

$$M_z = I_n(f_n(z \mid \cdot)), \qquad z \in T,$$

for some $n \geq 1$, where $f_n(z \mid \cdot)) \in L^2(T_n)$. Then it follows from (A3) that (see Lemma 5.4.1), for any $z' < z''$

$$f_n((z', z''] \mid \cdot) = 0, \text{ a.s.} \tag{5.2.2}$$

on the set $\{(z_1, \dots, z_n) \in T_n : \; z_i \notin (z', z''], \text{ for any } i = 1, \dots, n\}$.

For $z \in T$, put

$$A_{ij}(z) = \{(z_1, \dots, z_n) \in T_n : \; z \in [s_i, s_{i+1}) \times [t_{(j)}, t_{(j+1)})\}, \quad i, j = 0, 1, \cdots, n-1$$

$$A_{in}(z) = \{(z_1, \dots, z_n) \in T_n : \; z \in [s_i, s_{i+1}) \times [t_{(n)}, 1)\}, \quad i = 0, 1, \cdots, n-1$$

$$A_{nj}(z) = \{(z_1, \dots, z_n) \in T_n : \; z \in [s_n, 1] \times [t_{(j)}, t_{(j+1)})\}, \quad j = 0, 1, \cdots, n-1$$

$$A_{nn}(z) = \{(z_1, \dots, z_n) \in T_n : \; z \in [s_n, 1] \times [t_{(n)}, 1]\},$$

where $s_0 = 0$, and $0 = t_{(0)} \leq t_{(1)} < t_{(2)} < \cdots < t_{(n)} \leq t_{(n+1)} = 1$ is an ordering of $(t_1, \dots, t_n)$.

By (5.2.2),

$$f_n((z', z''] \mid \cdot) = 0, \text{ a.s. on the set } A_{ij}(z') \cap A_{ij}(z''), \tag{5.2.3}$$

for any $0 \leq i, j \leq n$, since if $(z_1, \dots, z_n) \in A_{ij}(z') \cap A_{ij}(z'')$, then all $z_k \notin (z', z'']$.

The sets $A_{ij}(z)$ satisfy (see Lemma 5.4.2)

(B1) $A_{00}(0) = T_n$, $A_{nn}(1) = T_n$, a.s.

(B2) $\bigcup_{0 \leq i, j \leq n} A_{ij}(z) = T_n$, a.s. for every $z \in T$.

($B3$) For any $i, j = 0, 1, \ldots, n$ and for any countable and dense subset B of T, we have:

$$\bigcup_{z \in B} A_{ij}(z) = T_n, \qquad \text{a.s.}$$

Define functions h_{ijkl}, $0 \le i \le k \le n$, $0 \le j \le l \le n$, on the set T_n by

$$h_{ijkl}(z) = f_n((z', z''] \mid z)$$

for some $z' \in Q \cap (s_i, s_{i+1}) \times (t_{(j)}, t_{(j+1)})$, and $z'' \in Q \cap (s_k, s_{k+1}) \times (t_{(l)}, t_{(l+1)})$ which are well-defined due to (5.2.3) (see Lemma 5.4.3). Here Q denotes the set of all points with rational coordinates. From (5.2.2),

$$h_{ijil} = h_{ijkj} = 0, \qquad l \ge j, \ k \ge i.$$

From (5.2.3) and (B3) it follows that (see Lemma 5.4.4)

$$f_n((z', z''] \mid \cdot) = h_{ijkl} \quad \text{a.s. on} \quad A_{ij}(z') \cap A_{kl}(z''), \quad \text{for all} \quad z' < z'', \tag{5.2.4}$$

so that from (B2), a.s. on T_n

$$f_n((z', z''] \mid \cdot) = \sum_{0 \le i < k \le n,\ 0 \le j < l \le n} h_{ijkl}(z) 1_{A_{ij}(z')}(\cdot) 1_{A_{kl}(z'')}(\cdot).$$

Thus,

$$I_n(f_n((z', z''] \mid \cdot)) = I_n\Big(\sum_{0 \le i < k \le n,\ o \le j < l \le n} h_{ijkl}(z) 1_{A_{ij}(z')}(\cdot) 1_{A_{kl}(z'')}(\cdot)\Big). \tag*{(5.2.4)'}$$

By the definition of h,

$$h_{ijkl} = \sum_{i \le i' < k,\ j \le j' < l} h_{i' j' i'+1\, j'+1}, \tag{5.2.5}$$

so for a.s. $z \in T_n$ (see Lemma 5.4.5)

$$f_n((z', z''] \mid \cdot) = \sum_{i<k,\, j<l} \; \sum_{i \le i' < k,\, j \le j' < l} h_{i'\ j'\ i'+1\ j'+1}(z) 1_{A_{ij}(z')}(z) 1_{A_{kl}(z'')}(z)$$

$$= \sum_{i', j'} h_{i'\ j'\ i'+1\ j'+1}(z) [\sum_{i \le i' < k,\, j \le j' < l} 1_{A_{ij}(z')}(z) 1_{A_{kl}(z'')}(z)]$$

$$= \sum_{i,j} h_{i\ j\ i+1\ j+1}(z) 1_{(z', z'']}(s_{i+1}, t_{(j+1)}). \tag{5.2.6}$$

If $t_{i+1} \neq t_{(j+1)}$, then for any $z' \in (s_i, s_{i+1}) \times (t_{(j)}, t_{(j+1)})$, $z'' \in (s_{i+1,}, s_{i+2}) \times (t_{(j+1)}, t_{(j+2)})$, we have that $z_k \notin (z', z'']$, for $k = 1, 2, \ldots n$. Then it follows from (5.2.2) that

$$h_{i\ j\ i+1\ j+1}(z) = 0 \qquad \text{if} \qquad t_{i+1} \neq t_{(j+1)}. \tag{5.2.7}$$

Therefore a.s. on T_n,

$$f_n((z', z''] \mid \cdot) = \sum_{i,\, j} h_{i\ j\ i+1\ j+1}(z) 1_{(z', z'']}(z_{i+1}). \tag{5.2.8}$$

To obtain functions g_n, we introduce the following notations: denote by $\sigma(k)$ a permutation of $(1, 2, \ldots, k-1)$, $k \ge 2$ and $\sigma(1) = \phi$. For $1 \le k \le n$, $1 \le i < k$ and $\sigma(k) = (\sigma_1, \sigma_2, \ldots, \sigma_{k-1})$, let

$$T_{k\ i\ \sigma(k)} = \{(z_1, \ldots, z_n) \in T_n :\ t_p = t_{(r_p)}, \text{ where } r_p = \sigma_p \text{ if } p \le i;\ r_i = k;$$

$$r_p = \sigma_{p-1} \text{ if } i < p < k;\ r_p = p, \text{ if } k < p \le n\},$$

$$T_{1\ 0\ \sigma(1)} = \{(z_1, \ldots, z_n) \in T_n :\ z_1 < \cdots < z_n\},$$

and $T_{k\ 0\ \sigma(k)} = \phi$ if $k \ge 2$. Then we have (see Lemma 5.4.6)

$$\bigcup_{1 \le k \le n,\ 0 \le i < k,\, \sigma(k)} T_{k\ i\ \sigma(k)} = T_n \qquad \text{a.s.} \tag{5.2.9}$$

It follows from (5.2.7) that

$$h_{p\ j\ p+1\ j+1} = 0, \quad \text{unless } j + 1 = r_{p+1} \quad \text{a.s. on the set } T_{k\ i\ \sigma(k)}. \tag{5.2.10}$$

For $1 \leq k \leq n$, $0 \leq i < k$ and $\sigma(k)$, we define functions g on the set $T \times T_{n-1}$ by

$$g_{k\ i\ \sigma(k)}(z \mid z_1, \dots, z_{n-1})$$

$$= \sum_{p=0}^{n-1} h_{p\ r_{p+1}-1\ p+1\ r_{p+1}}(z_1, \dots, z_p, z, z_{p+1}, \dots, z_{n-1})$$

$$\times 1\{(z_1, \dots, z_p, z, z_{p+1}, \dots, z_{n-1}) \in T_{k\ i\ \sigma(k)}\},$$

and

$$g_{n-1}(z \mid z_1, \dots, z_{n-1}) = \sum_{k\ i\ \sigma(k)} g_{k\ i\ \sigma(k)}(z \mid z_1, \dots, z_{n-1})$$

for $z \in T$ and $(z_1, \dots, z_{n-1}) \in T_{n-1}$.

Then (Lemma 5.4.8)

$$g_{n-1}(z_{p+1} \mid z_1, \dots, z_p, z_{p+2}, \dots, z_n) = h_{p\ r_{p+1}-1\ p+1\ r_{p+1}}(z_1, \dots, z_n) \tag{5.2.11}$$

on $T_{k\ i\ \sigma(k)}$, for $p = 0, 1, \dots, n-1$.

Therefore,

$$f_n((z', z''] \mid z) 1_{T_{k\ i\ \sigma(k)}}(z) = \sum_{p,j} h_{p\ j\ p+1\ j+1}(z) 1_{(z', z'']}(z_{p+1}) 1_{T_{k\ i\ \sigma(k)}}(z)$$

$$= \sum_{p=0}^{n-1} h_{p\ r_{p+1}-1\ p+1\ r_{p+1}}(z) 1_{(z', z'']}(z_{p+1}) 1_{T_{k\ i\ \sigma(k)}}(z)$$

$$= \sum_{p=0}^{n-1} g_{n-1}(z_{p+1} \mid z_1, \cdots, \hat{z}_{p+1}, \cdots, z_n) 1_{(z', z'']}(z_{p+1}) 1_{T_{k\ i\ \sigma(k)}}(z).$$

Hence

$$M(z', z''] = I_n(f_n((z', z''] \mid \cdot))$$

$$= \sum_{p=1}^{n} I_n(g_{n-1}(z_p \mid z_1, \cdots, \hat{z}_p, \cdots, z_n) 1_{(z', z'']}(z_p).$$

This is a Skorohod Representation for M_z by taking $z' = 0$, $z'' = z$, because of (A2). Notice that the function g_{n-1} is uniquely determined by f_n.

Conversely, suppose M possesses a Skorohod Representation. Recall the relation of the multiple integral [3]:

$$E(I_m(f_m) \mid \mathcal{F}_A) = I_m(g_m)$$

where $g_m(z_1, \ldots, z_m) = f_m(z_1, \ldots, z_m) 1_A(z_1) \cdots 1_A(z_m)$, we have the result immediately.

The proof can be extended to the general case with no difficulty.

5.3. The Skorohod Integral Representation

In this section we will consider the problem when a process can be expressed as an indefinite Skorohod stochastic integral.

For a process $M = \{M_z, z \in T\}$ such that $EM_z^2 < \infty$, for all z, we introduce the set

$$L(M) = \{ \sum_{0 \le p \le m, 0 \le q \le m'} (\Delta_{pq} M)^2 : 0 = s_0 \le s_1 < \cdots < s_m \le s_{m+1} = 1,\ m \ge 1;$$
$$0 = t_0 \le t_1 < \cdots < t_{m'} \le t_{m'+1} = 1,\ m' \ge 1 \}$$

where $\Delta_{pq} M = M(s_{p+1}, t_{q+1}) - M(s_p, t_{q+1}) - M(s_{p+1}, t_q) + M(s_p, t_q)$, and define

$$V^*(M) = \sup\{E\xi \,;\, \xi \in L(M)\}.$$

First we will consider a stochastic process of the form

$$M_z = I_n(f_n((z \mid \cdot)) \qquad z \in T,$$

belonging to $\mathcal{S}$. By the proof of Theorem 5.2.4, $f_n((z\,|\,\cdot)$ can be written as

$$f_n((z\,|\,z) = \sum_{i,\,j} h_{i\,j\,i+1\,j+1}(z)1_{R_z}(z_{i+1}) \tag{5.3.1}$$

$$= \sum_{p=1}^{n} g_{n\text{-}1}(z_p\,|\,z_1, \dots, \hat{z}_p, \dots, z_n)1_{R_z}(z_p), \tag{5.3.2}$$

for all $z \in T$.

LEMMA 5.3.1. *Let M belong to $\mathcal{S}$ and be of the form $M_z = I_n(f_n((z\,|\,\cdot))$, $z \in T$. Then M has an SIR, $M_z = \delta(u1_{R_z})$, if and only if the function $g_{n\text{-}1}$ in* (5.3.2) *is square integrable, which is equivalent to $h_{i\,j\,i+1\,j+1}$ being square integrable for all* $0 \leq i, j < n$. *And in this case*

$$E\int_T u_z^2\,dz = \sum_{i,\,j} \|h_{i\,j\,i+1\,j+1}\|^2 \tag{5.3.3}$$

where $\|\ \|$ denotes the norm on $L^2(T_n)$.

PROOF: Clearly M has an SIR if and only if the function $g_{n\text{-}1}$ is square integrable because of (5.3.2). In this case $M_z = \delta(u1_{R_z})$, with $u_z = I_{n\text{-}1}(g_{n\text{-}1}((z\,|\,\cdot))$. Thus

$$E\int_T u_z^2\,dz = E\int_T [I_{n\text{-}1}(g_{n\text{-}1}(z\,|\,\cdot))]^2dz = \int_T \Lambda_{n\text{-}1}(g_{n\text{-}1}^2(z\,|\,\cdot))dz\,,$$

where $\Lambda_n(g)$ denotes the integral of $g\colon T_n \to \mathbb{R}$ with respect to Lebesgue measure.

We will show that:

$$\int_T \Lambda_{n\text{-}1}(g_{n\text{-}1}^2(z\,|\,\cdot))dz = \sum_{i,\,j} \|h_{i\,j\,i+1\,j+1}\|^2\,.$$

In fact, the equality

$$g_{n-1}(z \mid z_1, \dots, z_{n-1}) = \sum_{k,\, i, \sigma(k)} \sum_{p=0}^{n-1} h_{p\; r_{p+1}-1\; p+1\; r_{p+1}}(z_1, \dots, z_p, z, z_{p+1}, \dots, z_{n-1})$$

$$\times 1\{(z_1, \dots, z_p, z, z_{p+1}, \dots, z_{n-1}) \in T_{k\; i\; \sigma(k)}\}$$

implies that

$$g_{n-1}^2(z \mid z_1, \dots, z_{n-1}) = \sum_{k,\, i, \sigma(k)} \sum_{p=0}^{n-1} h_{p-1\; p\; r_{p+1}-1\; p+1\; r_{p+1}}^2(z_1, \dots, z_p, z, z_{p+1}, \dots, z_{n-1})$$

$$\times 1\{(z_1, \dots, z_p, z, z_{p+1}, \dots, z_{n-1}) \in T_{k\; i\; \sigma(k)}\},$$

since all $T_{k\; i\; \sigma(k)}$ are disjoint. Thus from (5.2.10)

$$\int_T \Lambda_{n-1}(g_{n-1}^2(z \mid \cdot))\, dz$$

$$= \int_{T \times T_{n-1}} \{ \sum_{k,\, i, \sigma(k)} \sum_{p=0}^{n-1} h_{p-1\; p\; r_{p+1}-1\; p+1\; r_{p+1}}^2(z) 1_{T_{k\, i\, \sigma(k)}}(z) \} dz$$

$$= \sum_{i,j} \int_{T_n} h_{i\, j\, i+1\, j+1}^2(z) dz \;=\; \sum_{i,j} \|h_{i\; j\, i+1\, j+1}\|^2 .$$

THEOREM 5.3.2. *Let* $M = \{I_n(f_n((z \mid \cdot)), z \in T\}$ *belong to* $\mathcal{S}$. *Then* M *has an SIR,* $M_z = \delta(u 1_{R_z})$, *if and only if:*

(A4) $V(M) < +\infty$

and, in this case

$$E \int_T u_z^2\, dz = \lim_{|\Gamma| \to 0} E[\sum_{p,q} (\Delta_{pq} M)^2],$$

where

$$\Pi = \{0 = s_0 \le s_1 < \cdots < s_m \le s_{m+1} = 1,\ 0 = t_0 \le t_1 < \cdots < t_{m'} \le t_{m'+1} = 1\},$$

$m, m' \ge 1$ *and* $|\Pi| = max_{i\,j}\{(s_{i+1} - s_i), (t_{j+1} - t_j)\}$.

Moreover, if (A4) *does not hold, then*

$$lim\ sup_{|\Pi| \to 0}\, E[\sum_{p,q} (\Delta_{pq}M)^2] = +\infty.$$

PROOF: From (5.2.4)', we have

$$M(z', z''] = \sum_{0 \le i < k \le n,\ 0 \le j < l \le n} I_n(h_{ijkl}\, 1_{A_{ij}}(z')1_{A_{kl}}(z'')).$$

Now let

$$\Pi = \{0 = \hat{s}_0 \le \hat{s}_1 < \cdots < \hat{s}_m \le \hat{s}_{m+1} = 1,\ 0 = \hat{t}_0 \le \hat{t}_1 < \cdots < \hat{t}_{m'} \le \hat{t}_{m'+1} = 1\},$$

for some $m, m' \ge 1$. Then

$$E[\sum_{p,q} (\Delta_{pq}M)^2] = E\sum_{p,q}[\sum_{0 \le i < k \le n,\ 0 \le j < l \le n} I_n(h_{ijkl}\, 1_{A_{ij}}(z'_{pq})1_{A_{kl}}(z''_{pq}))]^2$$

where $z'_{pq} = (\hat{s}_p, \hat{t}_q)$, $z''_{p+1q+1} = (\hat{s}_{p+1}, \hat{t}_{q+1})$.

Thus

$$E[\sum_{p,q} (\Delta_{pq}M)^2] = \sum_{p,q} \sum_{i\,j\,k\,l} \Lambda_n(h^2_{ijkl}\, 1_{A_{ij}}(z'_{pq})1_{A_{kl}}(z''_{pq})).$$

Put

$A_{ijkl}(\Pi) = \{(z_1, \ldots, z_n) \in T_n :$ there exist $0 \le p \le m$, $0 \le q \le m'$, such that

$$s_i \le \hat{s}_p < s_{i+1},\ s_k \le \hat{s}_{p+1} < s_{k+1},\ t_{(j)} \le \hat{t}_q < t_{(j+1)},\ t_{(l)} \le \hat{t}_{q+1} < t_{(l+1)}\}$$

for $0 \le i \le k \le n$, $0 \le j \le l \le n$, where $s_0 = t_{(0)} = 0$, and $s_{n+1} = t_{(n+1)} = 1$.

Then

$$\bigcup_{p\,q} A_{ij}(z'_{pq}) \cap A_{ij}(z''_{pq}) = A_{ijkl}(\Pi), \quad \text{a.s.}$$

Therefore

$$E[\sum_{p,q} (\Delta_{pq}M)^2] = \sum_{ijkl} \Lambda_n(h^2_{ijkl}\, 1_{A_{ijkl}(\Pi)}). \tag{5.3.4}$$

Now suppose that M has an SIR. It follows from Lemma 5.2.2 and (5.2.5) that every h_{ijkl} is square integrable on T_n, thus

$$E[\sum_{p,q} (\Delta_{pq}M)^2] \le \sum_{ijkl} \Lambda_n(h^2_{ijkl}) < +\infty,$$

and the condition $(A4)$ holds.

Notice that $|\Pi|$ goes to zero, we have (see Lemma 5.4.8)

$$A_{ijkl}(\Pi) \to 0, \tag{5.3.5}$$

for $i \le k-2$ and $j \le l-1$ or $i \le k-1$ and $j \le l-2$, where $|\cdot|$ denotes Lebesgue measure. From (5.3.4) it follows that (see Lemma 5.4.9)

$$lim_{|\Pi| \to 0} E[\sum_{p,q} (\Delta_{pq}M)^2] = \sum_{i,j} \Lambda_n(h^2_{i\,j\,i+1\,j+1}). \tag{5.3.6}$$

Now suppose that $(A4)$ is true. We will show that M has an SIR. By Lemma 5.3.1, we need to show that all $h_{ij\,i+1\,j+1}$ are square integrable. From (5.3.4)

$$E[\sum_{p,q} (\Delta_{pq}M)^2] \ge \sum_{i,j} \Lambda_n(h^2_{i\,j\,i+1\,j+1}\, 1_{A_{i\,i+1\,i+1\,j+1}(\Pi)}). \tag{5.3.7}$$

Suppose that the partition Π' is finer than Π, then it holds that $A_{i\,j\,i+1\,j+1}(\Pi) \subset$

$A_{i\,j\,i+1\,j+1}(\Pi')$. Consequently, for any sequence of partitions $\{\Pi_{mm'}\,;\, m, m' \geq 1\}$ verifying

$$lim_{m,m'}\, |\Pi_{mm'}| = 0 \text{ and } \Pi_{mm'} \uparrow \text{ in } m \text{ and } m',$$

the right hand side of (5.3.7) converges to $\sum_{i,j} \Lambda_n(h^2_{i\,j\,i+1\,j+1})$. Therefore, this quantity will be finite if ($A4$) holds, and otherwise we will have

$$lim\ sup_{\,|\Pi| \to 0}\, E[\sum_{p,q} (\Delta_{pq} M)^2] = +\infty.$$

REMARK: Similar results to Propositions 2.2-2.5 of [1] for two-parameter processes can be given.

5.4. Appendix

LEMMA 5.4.1 *Proof of* (5.2.2):

$$f_n((z', z''] \mid \cdot) = 0, \ a.s. \quad for\ any\ z' < z''$$

on the set $\{(z_1, \ldots, z_n) \in T_n :\ z_i \notin (z', z''],\ for\ any\ i = 1, \ldots, n\}$.

PROOF: Recall the relation

$$E(I_m(f_m) \mid \mathcal{F}_A) = I_n(f_n(z_1, \ldots, z_n) 1_A(z_1) \cdots 1_A(z_n)) \tag{5.4.1}$$

for any $f_n \in L^2(T^n)$, $A \in \mathcal{B}(T)$ (see (3.1) of [3]).

Now, $M_z = I_n(f_n((z \mid \cdot)$. Putting $A = (z', z'']^c$, we have from (A3) and (5.4.1) that

$$0 = E[I_n(f_n((z', z''] \mid \cdot) \mid \mathcal{F}_A] = I_n(f_n(z_1, \ldots, z_n) 1_A(z_1) \cdots 1_A(z_n)),$$

so

$$E[I_n(f_n((z', z''] \mid z_1, \dots, z_n)1_A(z_1) \cdots 1_A(z_n))]^2$$

$$= n! \, \|f_n((z', z''] \mid z_1, \dots, z_n)1_A(z_1) \cdots 1_A(z_n))\|^2 = 0.$$

Therefore $f_n((z', z''] \mid z_1, \dots, z_n) = 0$ if all $z_i \notin (z', z'']$, $i = 1, \dots, n$.

LEMMA 5.4.2 *Proof of* (B1) - (B3): *The sets* $A_{ij}(z)$ *satisfy*:

($B1$) $A_{00}(0) = T_n$, $A_{nn}(1) = T_n$, *a.s.*

($B2$) $\bigcup_{0 \le i, j \le n} A_{ij}(z)$, *a.s.* *for every* $z \in T$.

($B3$) *For any* $i, j = 0, 1, \dots, n$ *and for any countable and dense subset* B *of* T, *we have:*

$$\bigcup_{z \in B} A_{ij}(z) = T_n, \qquad a.s.$$

PROOF: ($B1$) is obvious. ($B2$) follows from:

$$T_n = \{(z_1, \dots, z_n) \colon z_i = (s_i, t_i) \in T, i = 1, \dots, n, 0 \le s_1 < \cdots < s_0 \le 1,$$

$$0 \le t_{(1)} < t_{(2)} < \cdots < t_{(n)} \le 1\}$$

$$= \bigcup_{0 \le i, j \le n} A_{ij}(z), \qquad \text{a.s.}$$

($B3$) follows from the denseness of the B and the first equality above.

LEMMA 5.4.3. *Proof of the "well-defined" of* h_{ijkl}.

PROOF: First we will show that for a.s. $(z_1, \dots, z_n) \in T$, if $z, \overline{z} \in [s_i, s_{i+1}) \times [t_{(j)}, t_{(j+1)})$, then

$$f_n(\overline{z} \mid z) = f_n(z \mid z). \tag{5.4.2}$$

Let $z < \bar{z}$, then

$$f_n(\bar{z} \mid z) = f_n(z \mid z) + f_n(A_1 \mid z) + f_n(A_2 \mid z) + f_n(A_3 \mid z)$$

where $A_1 = [0, s]\times(t, \bar{t}]$, $A_2 = (z, \bar{z}]$, $A_3 = (s, \bar{s}]\times(0, t]$.

Note all $z_k \notin A_i$, $i = 1, 2, 3$. From (5.2.3) we have

$$f_n(A_i \mid z) = 0 \quad i = 1, 2, 3.$$

This is (5.4.2).

In general, let $z, \bar{z} \in [s_i, s_{i+1})\times[t_{(j)}, t_{(j+1)})$. Since $z_k \notin [0, s\vee\bar{s}]\times[0, t\vee\bar{t}]\backslash[0, s\wedge\bar{s}]\times[0, t\wedge\bar{t}]$, for all k, (5.4.2) follows.

Now define h_{ijkl}, $0 \le i \le k \le n$, $0 \le j \le l \le n$, on the set T_n by

$$h_{ijkl}(z) = f_n((z', z''] \mid z)$$

for some $z' \in Q\cap(s_i, s_{i+1})\times(t_{(j)}, t_{(j+1)})$, and $z'' \in Q\cap(s_k, s_{k+1})\times(t_{(l)}, t_{(l+1)})$. It remains to show

$$f_n((z', z''] \mid z) = f_n(\bar{z}', \bar{z}''] \mid z), \tag{5.4.3}$$

if $\bar{z}' \in Q\cap(s_i, s_{i+1})\times(t_{(j)}, t_{(j+1)})$, and $\bar{z}'' \in Q\cap(s_k, s_{k+1})\times(t_{(l)}, t_{(l+1)})$.

Note

$$z' = (s', t'),\quad \bar{z}' = (\bar{s}', \bar{t}') \in Q\cap(s_i, s_{i+1})\times(t_{(j)}, t_{(j+1)}),$$

$$z'' = (s'', t''),\quad \bar{z}'' = (\bar{s}'', \bar{t}'') \in Q\cap(s_k, s_{k+1})\times(t_{(l)}, t_{(l+1)}),$$

so

$$(s', t''),\ (\bar{s}', \bar{t}'') \in Q\cap(s_i, s_{i+1})\times t_{(l)}, t_{(l+1)}),$$

$$(s'', t'),\ (\bar{s}'', \bar{t}') \in Q \cap (s_k, s_{k+1}) \times (t_{(j)}, t_{(j+1)}).$$

Thus (5.4.3) follows from the following equalities:

$$\begin{aligned} f_n((z', z''] \mid z) &= f_n(z'' \mid z) - f_n((s'', t') \mid z) - f_n((s', t'') \mid z) + f_n(z' \mid z) \\ &= f_n(\bar{z}'' \mid z) - f_n((\bar{s}'', \bar{t}') \mid z) - f_n((\bar{s}', \bar{t}'') \mid z) + f_n(\bar{z}' \mid z) \\ &= f_n((\bar{z}', \bar{z}''] \mid z). \end{aligned}$$

Therefore h_{ijkl} is well-defined.

LEMMA 5.4.4. *Proof of* (5.2.4):

$$f_n((z', z''] \mid \cdot) = h_{ijkl} \quad \textit{a.s. on} \quad \mathrm{A}_{ij}(z') \cap A_{kl}(z''), \quad \textit{for all} \quad z' < z''.$$

PROOF: Let $z', z'' \in Q$. Then for a.s. $z = (z_1, \ldots, z_n) \in \mathrm{A}_{ij}(z') \cap A_{kl}(z'')$, we have

$$z' \in Q \cap (s_i, s_{i+1}) \times (t_{(j)}, t_{(j+1)}), \text{ and } \quad z'' \in Q \cap (s_k, s_{k+1}) \times (t_{(l)}, t_{(l+1)}).$$

By the definition of h,

$$f_n((z', z''] \mid \cdot) = h_{ijkl}.$$

Suppose z' or $z'' \notin Q$. By (B3),

$$\begin{aligned} \mathrm{A}_{ij}(z') &= \bigcup_{\alpha \in B} A_{ij}(z') \cap A_{ij}(\alpha) && \text{a.s.} \\ \mathrm{A}_{kl}(z') &= \bigcup_{\beta \in B} A_{kl}(z') \cap A_{kl}(\beta) && \text{a.s.} \end{aligned}$$

Therefore, for almost all $z = (z_1, \ldots, z_n) \in \mathrm{A}_{ij}(z') \cap A_{kl}(z'')$, there exist α, $\beta \in Q$ such that $z \in A_{ij}(z') \cap \mathrm{A}_{ij}(\alpha) \cap A_{kl}(z') \cap \mathrm{A}_{kl}(\beta)$. Note $z', \alpha \in Q \cap (s_i, s_{i+1}) \times (t_{(j)}, t_{(j+1)})$, and $z'', \beta \in Q \cap (s_k, s_{k+1}) \times (t_{(l)}, t_{(l+1)})$. By the proof of Lemma 5.4.3,

$$f_n((z', z''] \mid z) = f_n((\alpha, \beta] \mid z).$$

On the other hand, because $\alpha, \beta \in Q$, we have

$$f_n((\alpha, \beta] \mid z) = h_{ijkl} \ .$$

Therefore

$$f_n((z', z''] \mid z) = h_{ijkl}\,(z).$$

LEMMA 5.4.5. *Proof of* (5.2.6):

$$\sum_{i \le i' < k,\, j \le j' < l} 1_{A_{ij}(z')}(z)\, 1_{A_{kl}(z'')}(z) = 1_{(z', z'']}(s_{i'+1}, t_{(j'+1)})$$

for $z = (z_1, \dots, z_n) \in T_n,\ 0 \le i' < n,\ 0 \le j' < n.$

PROOF:

$$l.h.s. = \sum_{i \le i',\, j \le j'} 1_{A_{ij}(z')}(z) \sum_{k > i',\, l > j'} 1_{A_{kl}(z'')}(z)]$$

$$= 1_{\bigcup_{i \le i', j \le j'} A_{ij}(z)}(z)\ 1_{\bigcup_{k > i', l > j'} A_{kl}(z'')}(z).$$

Note

$$\bigcup_{i \le i',\, j \le j'} A_{ij}(z') = \{(z_1, \dots, z_n) \in T_n\colon\ z' \in [0, s_{i'+1}) \times [0,\ t_{(j'+1)})\},$$

$$\bigcup_{k > i',\, l > j'} A_{kl}(z'') = \{(z_1, \dots, z_n) \in T_n\colon\ z'' \in [s_{i'+1,\,1}] \times [t_{(j'+1)}, 1]\}.$$

So

$$\sum_{i \le i' < k,\, j \le j' < l} 1_{A_{ij}(z')}(z)\, 1_{A_{kl}(z'')}(z) = 1 \Leftrightarrow z' < (s_{i'+1}, t_{(j'+1)}) \le z''$$

$$\Leftrightarrow 1_{(z', z'']}(s_{i'+1}, t_{(j'+1)}) = 1.$$

LEMMA 5.4.6. *Proof of* (5.2.9):

$$\bigcup_{1\le k\le n,\ 0\le i<k,\sigma(k)} T_{k\ i\ \sigma(k)} = T_n \qquad a.s.$$

PROOF: Note

$$T_n = \{(z_1, \dots, z_n)\colon z_i = (s_i, t_i) \in T,\ i = 1, \dots, n,\ 0 \le s_1 < \cdots < s_n \le 1,$$

$$0 \le t_{(1)} < t_{(2)} < \cdots < t_{(n)} \le 1\}.$$

Consider $z = (z_1, \dots, z_n) \in T_n$ with $z_i = (s_i, t_i)$. If $t_i = t_{(i)}$, then $z \in T_{1\ 0\ \sigma(1)}$; otherwise we write $t_i = t_{(r_i)}$, $i = 1, 2, \dots, n$. Put $k = \min\{p' : p = r_p, p > p'\}$ and let $t_i = t_{(k)}$ for some $i < k$, then $z \in T_{k\ i\ \sigma(k)}$ with $\sigma(k) = (r_1, r_2, \dots, r_{k-1})$.

LEMMA 5.4.7. *Proof of* (5.2.11):

$$g_{n-1}(z_{p+1} \mid z_1, \dots, z_p, z_{p+2}, \dots, z_n) = h_{p\ r_{p+1}-1\ p+1\ r_{p+1}}(z_1, \dots, z_n)$$

on $T_{k\ i\ \sigma(k)}$, *for* $p = 0, 1, \dots, n-1$.

PROOF: Let $z = (z_1, \dots, z_n) \in T_{k\ i\ \sigma(k)}$. By the definition of g, we have

$$g_{n-1}(z_{p+1} \mid z_1, \dots, \hat{z}_{p+1}, \dots, z_n) = \sum_{k'\ i'\ \sigma(k')} g_{k'\ i'\ \sigma(k')}(z_{p+1} \mid z_1, \dots, \hat{z}_{p+1}, \dots, z_n)$$

$$= \sum_{k'\ i'\ \sigma(k')} \sum_{p'=0}^{n-1} h_{p'\ r_{p'+1}-1\ p'+1\ r_{p'+1}}(z_1, \dots, z_{p'}, z_{p+1}, z_{p'+1}, \dots, z_n)$$

$$\times 1_{\{z_1, \dots, z_{p'}, z_{p+1}, z_{p'+1}, \dots, z_n) \in T_{k'\ i'\ \sigma(k')}\}}$$

$$= \sum_{k'\ i'\ \sigma(k')} h_{p\ r_{p+1}-1\ p+1\ r_{p+1}}(z_1, \dots, z_n) \times 1_{\{z_1, \dots, z_n) \in T_{k'\ i'\ \sigma(k')}\}}$$

$$+ \sum_{k'\ i'\ \sigma(k')} \sum_{p'<p} + \sum_{k'\ i'\ \sigma(k')} \sum_{p'>p}$$

Note, the first term equals to $h_{p\, r_{p+1}-1\, p+1\, r_{p+1}}(z_1, \dots, z_n)$, and

$$(z_1, \dots, z_{p'}, z_{p+1}, z_{p'+1}, \dots, \hat{z}_{p+1}, \dots, z_n) \notin T_n \qquad \text{if} \quad p' < p,$$

$$(z_1, \dots, \hat{z}_{p+1}, \dots, z_{p'}, z_{p+1}, z_{p'+1}, \dots, z_n) \notin T_n \qquad \text{if} \quad p' > p,$$

so that the second and the third terms are zero.

LEMMA 5.4.8. *Proof of* (5.3.5): *As* $|\Pi|$ *goes to zero, we have*

$$A_{ijkl}(\Pi) \to 0,$$

for $i \leq k-2$ *and* $j \leq l-1$ *or* $i \leq k-1$ *and* $j \leq l-2$.

PROOF: If $i \leq k-1$ and $j \leq l-2$, we have

$$\begin{aligned} A_{ijkl}(\Pi) &= \bigcup_{p\,q} \{z = (z_1, \dots, z_n) \in T_n : s_i \leq \hat{s}_p < s_{i+1},\ s_k \leq \hat{s}_{p+1} < s_{k+1}, \\ &\qquad t_{(j)} \leq \hat{t}_q < t_{(j+1)},\ t_{(l)} \leq \hat{t}_{q+1} < t_{(l+1)}\} \\ &\subset \bigcup_{p\,q} \{z \in T_n : s_{i+1} \in (\hat{s}_p, \hat{s}_{p+1}],\ t_{(j+1)}, t_{(l)} \in (\hat{t}_q, \hat{t}_{q+1}]\} \\ &\subset \bigcup_{p\,q} \bigcup_{(r, r') \sim (1, 2, \dots, n-1)} \{z \in T_n : s_{i+1} \in (\hat{s}_p, \hat{s}_{p+1}],\ t_r,\ t_{r'} \in (\hat{t}_q, \hat{t}_{q+1}]\}, \end{aligned}$$

where $(r, r') \sim (1, 2, \dots, n-1)$ means a combination of n-1 taken 2. So

$$|A_{ijkl}(\Pi)| \leq \sum_{p,q} C^2_{n-1}(\hat{s}_{p+1} - \hat{s}_p)(\hat{t}_q, \hat{t}_{q+1})^2$$

$$\leq n(n-1)|\Pi| \sum_{p,q} (\hat{s}_{p+1} - \hat{s}_p)(\hat{t}_q - \hat{t}_{q+1}) = n(n-1)|\Pi| \to 0,$$

as $|\Pi| \to 0$.

The same argument will be applied to the case $i \leq k-2$ and $j \leq l-1$.

LEMMA 5.4.9. *Proof of* (5.3.6):

$$lim_{|\Pi| \to 0} E[\sum_{p,q} (\Delta_{pq}M)^2] = \sum_{i,j} \Lambda_n(h^2_{i\,j\,i+1\,j+1}).$$

PROOF: Note Π is a partition of T. We have

$$T_n = \{(z_1, \dots, z_n): z_i = (s_i, t_i) \in T, i = 1, \dots, n, 0 = s_0 \le s_1 < \cdots < s_n \le s_{n+1} = 1,$$
$$0 = t_{(0)} \le t_{(1)} < t_{(2)} < \cdots < t_{(n)} \le t_{(n+1)} = 1\}$$

$$= \bigcup_{p\,q} \bigcup_{k \ge i,\, l \ge j} \{(z_1, \dots, z_n) \in T_n : s_i \le \hat{s}_p < s_{i+1},\ s_k \le \hat{s}_{p+1} < s_{k+1},$$
$$t_{(j)} \le \hat{t}_q < t_{(j+1)},\ t_{(l)} \le \hat{t}_{q+1} < t_{(l+1)}\}$$

$$= \bigcup_{k \ge i,\, l \ge j} A_{ijkl}(\Pi)$$

$$= \bigcup_{i\,j} A_{ij i+1\, j+1}(\Pi) + \bigcup_{ijkl \in C} A_{ijkl}(\Pi) \qquad (5.4.4)$$

where

$$C = \{(i, j, k, l): i = k, l \ge j \ or \ i \le k, j = l \ or \ i \le k\text{-}2, j \le l\text{-}2$$
$$or\ i = k\text{-}1, j \le l\text{-}2\ or\ i \le k\text{-}2, j = l\text{-}1\}.$$

Since $h_{ijij} = h_{ijil} = h_{ijkj} = 0$, for $l \ge j, k \ge i$, we have

$$\sum_{i\,j\,kl} \Lambda_n(h^2_{i\,j\,kl} 1_{A_{ijkl}(\Pi)}) = \sum_{i\,j} \Lambda_n(h^2_{i\,j\,i+1\,j+1} 1_{A_{i\,j\,i+1\,j+1}(\Pi)})$$

$$+ \Big(\sum_{k \ge i+2,\ l \ge j+2} + \sum_{k = i+1,\ l \ge j+2} + \sum_{k \ge i+2,\ l = j+1} \Big) \Lambda_n(h^2_{i\,j\,kl} 1_{A_{ijkl}(\Pi)})$$

$$= \sum_{ij} \Lambda_n(h^2_{i\,j\,i+1\,j+1})$$

$$+ \Big(\sum_{k \geq i+2,\ l \geq j+2} + \sum_{k=i+1,\ l \geq j+2} + \sum_{k \geq i+2,\ l=j+1} \Big) \Lambda_n((h^2_{i\,j\,kl} - h^2_{i\,j\,i+1\,j+1}) \mathbf{1}_{A_{ijkl}(\Pi)}).$$

By (5.3.5), the second part tends to zero. The result follows from (5.3.4).

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Chapter 6

LINEAR SKOROHOD STOCHASTIC DIFFERETIAL EQUATIONS IN THE TWO-PARAMETER CASE

1. Introduction

The Skorohod stochastic differential equations have been recently studied by several authors ([1]-[4], [6], [7], among others). Using Girsanov's transformation and Itô's formula, Buckdahn [2] considered the equation with Skorohod integral:

$$X_t = H + \int_0^t b(X_s)ds + \int_0^t a(X_s)dW_s, \quad \text{a.e. on } A \tag{6.1.1}$$

where A is a bounded ball in Ω. For a linear Skorohod stochastic differential equation:

$$X_t = H + \int_0^t A_s X_s ds + \int_0^t B_s X_s dW_s, \tag{6.1.2}$$

Shiota [6] constructed a solution by means of the Wiener-Itô chaos decomposition, when (A_s) and (B_s) are deterministic processes and H is a random variable represented by the finite sum of multiple Wiener integrals. Using Girsanov's transformation, Buckdahn [1] solved the equation under rather weak assumptions, which allow A and B to anticipate the future of the driving Wiener process.

To extend above results to the two-parameter case, we face some technical difficulties. For example: a) Unlike the one-parameter case, the solution of a linear system with nonanticipating initial condition cannot be expressed as an exponential. This property which we feel is required when one applies Girsanov's transformation in the one-parameter case. b) For a one-parameter process of the form

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dW_s,$$

a twice continuously differential function $f(X_t)$ is again expressible as the sum of a stochastic integral and an ordinary integral. However, when we consider the two-parameter case:

$$X_z = X_0 + \int_{R_z} \phi_\zeta \, d\zeta + \int_{R_z} \psi_\zeta dW_\zeta,$$

the Itô formula [8] decomposes function $f(X_z)$ as a sum of six integrals, which includes mixed integral and multiple integral. It is therefore not easy to adapt Buckdahn's technique to the two-parameter case.

In this chapter we construct a solution of the two-parameter linear equation with anticipating initial condition:

$$X_z = H + \int_{R_z} (A_\zeta X_\zeta + a_\zeta) d\zeta + \int_{R_z} (B_\zeta X_\zeta + b_\zeta) dW_\zeta, \qquad (1.3)$$

using the method of the Wiener chaos decomposition ([6]), where H has a finite chaos expansion.

6.2. Uniqueness of the Solution

In this section we give a proof of the uniqueness of the solution following the lines of Shiota [6] in the one parameter case.

Consider the linear Skorohod integral equation:

$$X_z = H + \int_{R_z} (A_\zeta X_\zeta + a_\zeta) d\zeta + \int_{R_z} (B_\zeta X_\zeta + b_\zeta) dW_\zeta \qquad (6.2.1)$$

where as above we assume that the initial random variable H has a finite chaos expansion and A, a, B, b are bounded deterministic functions. A process $(X_z) \in L^2(T\times\Omega)$ is called a solution of (6.2.1) if it satisfies (6.2.1) in $L^2(T\times\Omega)$. By the uniqueness of the solution (6.2.1) we understand the uniqueness in the class $L^2(T\times\Omega)$.

For $p = 1, 2, \dots$, let

$$\mathcal{H}_p(\Omega) = \{F \in L^2(\Omega):\ F = I_p(f_p),\ f_p \in \hat{L}^2(T^p)\},$$

$$\mathcal{H}_p(T\times\Omega) = \{u \in L^2(T\times\Omega):\ u_z = I_p(f_p(z,\cdot)), f_p \in L^2(T^{p+1}), f_p(z,\cdot) \in \hat{L}^2(T^p), \forall\ z \in T\}.$$

For convenience, we set $\mathcal{H}_0(T\times\Omega) = L^2(T\times\Omega)$, $\mathcal{H}_0(\Omega)$ = the constant system. By the Wiener-Itô decomposition

$$L^2(\Omega) = \sum_{p=0}^{\infty} \oplus \mathcal{H}_p\ (\Omega),$$

$$L^2(T\times\Omega) = \sum_{p=0}^{\infty} \oplus \mathcal{H}_p\ (T\times\Omega),$$

where $\oplus$ denotes the direct sum. We also define

$$\mathcal{H}_{finite}(\Omega) = \bigcup_{n\geq 0} \sum_{p=0}^{n} \oplus \mathcal{H}_p\ (\Omega),$$

$$\mathcal{H}_{finite}(T\times\Omega) = \bigcup_{n\geq 0} \sum_{p=0}^{n} \oplus \mathcal{H}_p\ (T\times\Omega).$$

PROPOSITION 6.2.1. *There exists at most one solution of* (6.2.1).

PROOF: It suffices to show that the equation below (6.2.2) does not have a nontrival solution:

$$Y_z = \int_{R_z} A_\zeta Y_\zeta d\zeta + \int_{R_z} B_\zeta Y_\zeta dW_\zeta, \tag{6.2.2}$$

where Y is the difference of two solutions of (6.2.1).

Since $Y \in L^2(T\times\Omega)$, we can write

$$Y_z = \sum_{p=0}^{\infty} Y_p(z),$$

where $Y_p \in \mathcal{H}_p\,(T\times\Omega)$, $p = 0, 1, \ldots$. Taking projections on $\mathcal{H}_p\,(T\times\Omega)$ of both sides of (6.2.2), we have

$$Y_0(z) = \int_{R_z} A_\zeta\, Y_0(\zeta)\, d\zeta\,,$$

$$Y_p(z) = \int_{R_z} A_\zeta\, Y_p(\zeta)\, d\zeta \;+\; \int_{R_z} B_\zeta\, Y_{p-1}(\zeta) dW_\zeta\,, \tag{6.2.3}$$

$p = 1, 2, \ldots$. By the assumption

$$Y_0(z)^2 = \Big(\int_{R_z} A_\zeta\, Y_0(\zeta)\, d\zeta\Big)^2 \le \text{const.} \int_{R_z} Y_0(\zeta)^2\, d\zeta\,.$$

Then it follows from Gronwall's inequality that $Y_0(z) = 0$, for every z. Hence the equation of (6.2.3) for $p = 1$ becomes

$$Y_1(z) = \int_{R_z} A_\zeta\, Y_1(\zeta)\, d\zeta\,.$$

Proceeding inductively, we have

$$\| Y \|^2_{L^2(T\times\Omega)} = \sum_{p=0}^{\infty} \| Y_p \|^2_{L^2(T\times\Omega)} = 0.$$

The proof of the proposition is complete.

6.3. Existence of the Solution

LEMMA 6.3.1. *The stochastic integral equation*

$$M_z = 1 + \int_{R_z} A_\zeta M_\zeta d\zeta \;+\; \int_{R_z} B_\zeta M_\zeta dW_\zeta \tag{6.3.1}$$

has a unique predictable solution such that M *belongs to* $\mathcal{H}^{(\infty)}(T\times\Omega) \equiv \bigcap_{n\ge 0} \mathcal{H}^{(n)}(T\times\Omega)$.

PROOF: The unique predictable solution M of (6.3.1) exists [9] which is the limit of the sequence $\{M_n\}$ given by

$$M_0(z) = 1,$$

$$M_{n+1}(z) = 1 + \int_{R_z} A_\zeta M_n(\zeta)\, d\zeta + \int_{R_z} B_\zeta M_n(\zeta) dW_\zeta .$$

By the iteration scheme,

$$M_n(z) = 1 + \sum_{k=1}^{n} \sum_{\substack{j_i = 0,1 \\ 1 \le i \le k}} \int_{\{0<z_1< \ldots < z_{k-1}< z_k<z\}} C_{j_1}(z_1) \cdots C_{j_{k-1}}(z_{k-1}) C_{j_k}(z_k)$$

$$dW^{j_1}(z_1) \cdots dW^{j_{k-1}}(z_{k-1}) dW^{j_k}(z_k)$$

where $C_0(z) = A_z$, $B_1(z) = B_z$, $dW^0(z) = dz$, $dW^1(z) = dW_z$.

Set

$$\mathcal{L} = \{\alpha = (j_1, j_2, \ldots, j_k): \; j_i \in \{0, 1\}, \; i = 1, \ldots, k, \; k = 1, 2, \ldots \}.$$

For $\alpha = (j_1, j_2, \ldots, j_k) \in \mathcal{L}$, denote $l(\alpha) = k$, $|\alpha| = \sum_{i=1}^{k} j_i$.

Let $\alpha = (j_1, j_2, \ldots, j_k)$, $\beta = (m_1, m_2, \ldots, m_{k'}) \in \mathcal{L}$. It follows from Proposition 6.4.4 in the Appendix that for

$$P_z^{\alpha} \equiv \int_{\{0<z_1< \ldots < z_{k-1}< z_k<z\}} C_{j_1}(z_1) \cdots C_{j_{k-1}}(z_{k-1}) C_{j_k}(z_k) dW^{j_1}(z_1) \cdots dW^{j_{k-1}}(z_{k-1}) dW^{j_k}(z_k),$$

we have

$$E[P_z^{\alpha} P_z^{\beta}] = 0, \qquad \text{if } |\alpha| \neq |\beta| \tag{6.3.2}$$

and

$$E[P_z^{\alpha} P_z^{\beta}] \le |C|^{k+k'} (C_{k+k'-2p}^{k-p})^2 \frac{1}{((k+k'-p)!)^2} \qquad \text{if } |\alpha| = |\beta| = p \tag{6.3.3}$$

where $|C| = max_i \, sup_{z\in T}|C_i(z)|$, and

$$C_m^r = \frac{m!}{r!(m-r)!}$$

for $r, m = 0, 1, \ldots,$ and $r \le m$.

Write

$$M_n(z) = 1 + \sum_{k=1}^{n} \sum_{l(\alpha)=k} P_z^{\alpha}$$

$$= (1 + \sum_{k=1}^{n} \sum_{l(\alpha)=k,\, |\alpha|=0} P_z^{\alpha}) + \sum_{k=1}^{n} \sum_{p=1}^{k} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha}$$

$$= (1 + \sum_{k=1}^{n} \sum_{l(\alpha)=k,\, |\alpha|=0} P_z^{\alpha}) + \sum_{p=1}^{n} \sum_{k=p}^{n} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha} .$$

For $m \ge n$, consider:

$$E(\sum_{p=1}^{m} \sum_{k=p}^{m} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha} - \sum_{p=1}^{n} \sum_{k=p}^{n} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha})^2$$

$$\le 2\, E(\sum_{p=n+1}^{m} \sum_{k=p}^{m} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha})^2 + 2\, E(\sum_{p=1}^{n} \sum_{k=n+1}^{m} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha})^2.$$

By (6.3.2) and (6.3.3) we have

$$E(\sum_{p=n+1}^{m} \sum_{k=p}^{m} \sum_{l(\alpha)=k,\, |\alpha|=p} P_z^{\alpha})^2$$

$$= \sum_{p=n+1}^{m} \sum_{k=p}^{m} \sum_{k'=p}^{m} \sum_{l(\alpha)=k,\ |\alpha|=p} \sum_{l(\beta)=k',\ |\beta|=p} E[\ P_z^{\alpha} P_z^{\beta}\ ]$$

$$\leq \sum_{p=n+1}^{m} \sum_{k=p}^{m} \sum_{k'=p}^{m} \sum_{l(\alpha)=k,\ |\alpha|=p} \sum_{l(\beta)=k',\ |\beta|=p} |C|^{k+k'} (C_{k+k'-2p}^{k-p})^2 \frac{1}{((k+k'-p)!)^2}$$

$$\leq \sum_{p=n+1}^{\infty} \sum_{k=p}^{\infty} \sum_{k'=p}^{\infty} \frac{|C|^{k+k'}}{[p!(k-p)!(k'-p)!]^2}$$

$$\leq \frac{1}{(n+1)!}\ exp\{2|C| + |C|^2\} \to 0,$$

as $n \to \infty$.

Similarly

$$E(\sum_{p=1}^{n} \sum_{k=n+1}^{m} \sum_{l(\alpha)=k,\ |\alpha|=p} P_z^{\alpha}\ )^2$$

$$\leq \sum_{p=1}^{n} \sum_{k=n+1}^{m} \sum_{k'=n+1}^{m} \frac{|C|^{k+k'}}{[p!(k-p)!(k'-p)!]^2}$$

$$\leq 2 \sum_{p=1}^{n} \sum_{k=n+1}^{m} \sum_{k'=n+1}^{m} \frac{|C|^{k+k'}}{(p!)^2 2^{k-p}(k-p)![(k'-p)!]^2}$$

$$= 2 \sum_{p=1}^{n} \frac{|C|^{2p}}{p!} \frac{2^p}{p!} \sum_{k=n+1}^{m} \frac{1}{2^k} \frac{|C|^{k-p}}{(k-p)!} \sum_{k'=n+1}^{m} \frac{|C|^{k'-p}}{[(k'-p)!]^2}$$

$$\leq \frac{4}{2^n} \sum_{p=1}^{n} \frac{|C|^{2p}}{p!} \sum_{k=p}^{m} \frac{|C|^{k-p}}{(k-p)!} \sum_{k'=p}^{m} \frac{|C|^{k'-p}}{[(k'-p)!]^2}$$

$$\leq \frac{4}{2^n} \exp\{\ |C|^2 + 2|C|\ \} \to 0,$$

as $n \to \infty$.

Hence

$$M_z = \sum_{p=0}^{\infty} F_p(z),$$

with

$$F_0(z) = 1 + \sum_{k=1}^{\infty} \sum_{l(\alpha)=k,\ |\alpha|=0} P_z^{\alpha}$$

$$F_p(z) = \sum_{k=p}^{\infty} \sum_{l(\alpha)=k,\ |\alpha|=p} P_z^{\alpha}.$$

For each $p \geq 1$, we have from (6.3.3) that,

$$E(F_p(z))^2 \leq \sum_{k=p}^{\infty} \sum_{k'=p}^{\infty} |C|^{k+k'} C_k^p C_{k'}^p (C_{k+k'-2p}^{k-p})^2 \frac{1}{((k+k'-p)!)^2}$$

$$= \sum_{k=p}^{\infty} \sum_{k'=p}^{\infty} |C|^{k+k'} C_k^p C_{k'}^p \left[\frac{(k+k'-2p)!}{(k-p)!(k'-p)!(k+k'-p)!}\right]^2$$

$$\leq \sum_{k=p}^{\infty} \sum_{k'=p}^{\infty} |C|^{k+k'} \frac{1}{p!(k-p)!(k'-p)!}.$$

Thus

$$\|M\|^2_{\mathcal{H}^{(m)}(T\times\Omega)} = \sum_{j=0}^{m} \sum_{p=j}^{\infty} \frac{p!}{(p-j)!} E\int_T (F_p(z))^2 dz$$

$$\leq \sum_{j=0}^{m} \sum_{p=j}^{\infty} \frac{p!}{(p-j)!} \sum_{k=p}^{\infty} \sum_{k'=p}^{\infty} |C|^{k+k'} \frac{1}{p!(k-p)!(k'-p)!}$$

$$= \sum_{j=0}^{m} |C|^{2j} \sum_{p=j}^{\infty} \frac{|C|^{2(p-j)}}{(p-j)!} \sum_{k=p}^{\infty} \frac{|C|^{k-p}}{(k-p)!} \sum_{k'=p}^{\infty} \frac{|C|^{k'-p}}{(k'-p)!}$$

$$= \sum_{j=0}^{m} |C|^{2j} \exp\{C^2 + 2C\} < \infty,$$

for every nonnegative integer m. Therefore M is in $\mathcal{H}^{(\infty)}(T\times\Omega)$.

We now construct a solution of the equation:

$$X_z = H + \int_{R_z} A_\zeta X_\zeta d\zeta + \int_{R_z} B_\zeta X_\zeta dW_\zeta \qquad (6.3.4)$$

for $H \in \mathcal{H}_p(\Omega)$.

THEOREM 6.3.2. *Let* $H \in \mathcal{H}_p(\Omega)$. *Then the equation* (6.3.4) *has a unique solution given by*

$$X_z = HM_z + \sum_{k=1}^{p} \frac{(-1)^k}{k!} <D^k H,\ D^k M_z>_{L^2(T^k)}, \qquad (6.3.5)$$

where M *is the solution of* (6.3.1).

PROOF: By Lemma 6.3.1, $M \in \mathcal{H}^{(\infty)}(T\times\Omega)$. Since $H \in \mathcal{H}_p(\Omega)$, it follows from Corollary 2.18 of [6] that $HM \in \mathcal{H}^{(\infty)}(T\times\Omega)$, $<D^k H,\ D^k M>_{L^2(T^k)} \in \mathcal{H}^{(\infty)}(T\times\Omega)$. Therefore the stochastic integrals appearing in (6.3.4) make sense.

By Proposition 1.2.6 we have

$$(D^k M_z)(r_1, r_2, \dots, r_k)$$
$$= \int_{R_z} A_\zeta (D^k M_\zeta)(r_1, r_2, \dots, r_k) d\zeta$$
$$+ \int_{R_z} B_\zeta (D^k M_\zeta)(r_1, r_2, \dots, r_k) dW_\zeta + \sum_{i=1}^{k} B_{r_i} D^{k-1} M_{r_i}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_k)$$

for $k \geq 1$, $r_i \in R_z$, $i = 1, \dots, k$, where $D^0 M_r = M_r$.

Then form Proposition 1.2.7

$$\int_{R_z^k} (D^kH)(\underline{r})(D^kM_z)(\underline{r})d\underline{r}$$

$$= \int_{R_z^k} (D^kH)(\underline{r})\{\int_{R_z} A_\zeta (D^kM_\zeta)(\underline{r})d\zeta + \int_{R_z} B_\zeta(D^kM_\zeta)(\underline{r})dW_\zeta$$

$$+ \sum_{i=1}^{k} B_{r_i}D^{k-1}M_{r_i}(r_1, \ldots , r_{i-1}, r_{i+1}, \ldots , r_k)\}d\underline{r}$$

$$= \int_{R_z^k} \int_{R_z} A_\zeta (D^kH)(\underline{r})(D^kM_\zeta)(\underline{r})d\zeta d\underline{r}$$

$$+ \int_{R_z^k} \int_{R_z} B_\zeta (D^kH)(\underline{r})(D^kM_\zeta)(\underline{r})dW_\zeta \, d\underline{r}$$

$$+ \int_{R_z^k} \int_{R_z} B_\zeta (D^{k+1}H)(\underline{r}, \zeta)(D^kM_\zeta)(\underline{r})d\zeta \, d\underline{r}$$

$$+ \; k\int_{R_z^k} B_{r_k} (D^kH)(\underline{r})(D^{k-1}M_{r_k})(r_1, \ldots , r_{k-1})d\underline{r}.$$

It follows from the adaptedness of M that

$$(D^kM_z)(\underline{r}) = 0, \text{ unless } \underline{r} \in [0, z]^k.$$

Then by Proposition 1.2.4

$$\sum_{k=1}^{p} \frac{(-1)^k}{k!} \langle D^kH,\, D^kM_z\rangle_{L^2(T^k)}$$

$$= \sum_{k=1}^{p} \frac{(-1)^k}{k!} \int_{R_z^k} (D^kH)(\underline{r})(D^kM_z)(\underline{r})d\underline{r}$$

$$= \sum_{k=1}^{p} \frac{(-1)^k}{k!}\int_{R_z} \int_{[0,\zeta]^k} A_\zeta(D^kH)(\underline{r})(D^kM_\zeta)(\underline{r})d\underline{r}d\zeta$$

$$+ \sum_{k=1}^{p} \frac{(-1)^k}{k!}\int_{R_z} \int_{[0,\zeta]^k} B_\zeta(D^kH)(\underline{r})(D^kM_\zeta)(\underline{r})d\underline{r}dW_\zeta$$

$$+ \sum_{k=1}^{p} \frac{(-1)^k}{k!}\int_{R_z} \int_{[0,\zeta]^k} B_\zeta(D^{k+1}H)(\underline{r}, \zeta)(D^kM_\zeta)(\underline{r})d\underline{r}d\zeta$$

$$+ \sum_{k=1}^{p} \frac{(-1)^k}{k!} k \int_{R_z^k} B_{r_k} (D^k H)(\underline{r})(D^{k-1} M_{r_k})(r_1, \dots, r_{k-1}) d\underline{r}$$

Note $D^{p+1}H = 0$, so that

$$\sum_{k=1}^{p} \frac{(-1)^k}{k!} \langle D^k H, D^k M_z \rangle_{L^2(T^k)}$$

$$= \sum_{k=1}^{p} \frac{(-1)^k}{k!} \int_{R_z} \int_{[0,\zeta]^k} A_\zeta (D^k H)(\underline{r})(D^k M_\zeta)(\underline{r}) d\underline{r} d\zeta$$

$$+ \sum_{k=1}^{p} \frac{(-1)^k}{k!} \int_{R_z} \int_{[0,\zeta]^k} B_\zeta (D^k H)(\underline{r})(D^k M_\zeta)(\underline{r}) d\underline{r} dW_\zeta$$

$$- \int_{R_z} B_\zeta D_\zeta H M_\zeta d\zeta \qquad (6.3.6)$$

Now let X be given by (6.3.5). From (6.3.6) we have

$$H + \int_{R_z} A_\zeta X_\zeta d\zeta + \int_{R_z} B_\zeta X_\zeta dW_\zeta$$

$$= H + \int_{R_z} A_\zeta H M_\zeta d\zeta + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \int_{R_z} \int_{[0,\zeta]^k} A_\zeta (D^k H)(\underline{r})(D^k M_\zeta)(\underline{r}) d\underline{r} d\zeta$$

$$+ \int_{R_z} B_\zeta H M_\zeta dW_\zeta + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \int_{R_z} \int_{[0,\zeta]^k} B_\zeta (D^k H)(\underline{r})(D^k M_\zeta)(\underline{r}) d\underline{r} dW_\zeta$$

$$= H M_z + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \int_{R_z^k} (D^k H)(\underline{r})(D^k M_z)(\underline{r}) d\underline{r}$$

$$= H M_z + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \langle D^k H, D^k M_z \rangle_{L^2(T^k)}.$$

The proof is complete.

COROLLARY 6.3.3. *If* $H = \sum_{p=1}^{n} H_p \in \mathcal{H}_{finite}(\Omega)$, *then the solution of* (6.3.4) *is given by*

$$X_z = HM_z + \sum_{p=1}^{n} \sum_{k=1}^{p} \frac{(-1)^k}{k!} <D^k H_p,\ D^k M_z>_{L^2(T^k)}$$

$$= HM_z + \sum_{k=1}^{n} \frac{(-1)^k}{k!} <D^k H,\ D^k M_z>_{L^2(T^k)}.$$

As in Shiota [6], we have:

COROLARY 6.3.4. *Let* $H \in \mathcal{H}_{finite}(\Omega)$. *Then the unique solution of* (6.2.1) *is given by*

$$Yz = X_z + N_z - M_z$$

where X_z *is given by* (6.3.5), M_z *is the solution of* (6.3.1), *and* N_z *is the solution of the following equation:*

$$N_z = 1 + \int_{R_z} (A_\zeta N_\zeta + a_\zeta) d\zeta + \int_{R_z} (B_\zeta N_\zeta + b_\zeta) dW_\zeta$$

6.4. Appendix

The aim of this section is to show the inequality (6.3.3). In the sequel we use the notations of [5].

Let

$$\mathcal{M} = \{ \alpha = (j_1, j_2, \dots, j_k) \colon \ j_i \in \{0, 1\} \ i = 1, \dots, k,\ k = 1, 2, \dots \} \cup \{v\}$$

denote the set of row vectors $\alpha = (j_1, j_2, \dots, j_k)$ with finite length $l(\alpha) \equiv k$, where $l(v) \equiv 0$. We write $-\alpha$ or $\alpha-$ if we delete the first or the last component of $\alpha \in \mathcal{M}$ ($l(\alpha) \geq 1$), respectively.

For $\alpha \in \mathcal{M}$, $z \in T$, we define recursively multiple stochastic integrals

$$I_\alpha(z) = \begin{cases} 1 & \text{for } \alpha = v \\ \int_{R_z} I_{\alpha-}(u) dW^{j_k}(u) & \text{for } \alpha = (j_1, j_2, \dots, j_k) \end{cases} \qquad (6.4.1)$$

where $dW^0(u) = du$, $dW^1(u) = dW_u$.

Then for $\alpha = (j_1, j_2, \dots, j_k) \in \mathcal{M}\backslash\{v\}$, we have

$$I_\alpha(z) = \int_{\{0<z_1<\dots<z_{k-1}<z_k<z\}} dW^{j_1}(z_1)dW^{j_2}(z_2)\cdots dW^{j_k}(z_k).$$

For $\alpha \in \mathcal{M}$, denote by α^* the vector which is obtained by deleting all components of α equal to 0. Let $\alpha \in \mathcal{M}$ satisfy that $\alpha = \alpha^*$ and $l(\alpha^*) = l$. For $l \in \{1, 2, \dots\}$, $k_i = \{0, 1, \dots\}$, $i \in \{0, 1, \dots, l\}$, and $z = (s, t) \in T$, we define recursively

$$H_{\alpha^*}(k_0, \dots, k_l, z)$$

$$= \begin{cases} 1 & \text{for } l = 0, k_0 = 0 \\ \int_{R_z} H_v(k_0 - 1, u)du & \text{for } l = 0, k_0 \geq 1 \\ \int_{R_z} \frac{(s-u_1)^{k_l}}{k_l\,!} \frac{(t-u_2)^{k_l}}{k_l\,!} H_{\alpha^{*}-}(k_0, \dots, k_{l-1}, u)dW_u & \text{for } l \geq 1 \end{cases} \qquad (6.4.2)$$

LEMMA 6.4.1.

$$\int_{R_z} H_{\alpha^*}(k_0, \dots, k_{l-1}, k_l, u)du = H_{\alpha^*}(k_0, \dots, k_{l-1}, k_l + 1, z).$$

PROOF: We consider two cases.

(1) $l = 0$.

Note $\alpha^* = v$. The result follows from (6.4.2).

(2) $l \geq 1$.

Write $R \equiv \int_{R_z} H_{\alpha^*}(k_0, \dots, k_{l-1}, k_l, u)du$.

By (6.4.2) and Proposition 1.2.5,

$$R = \int_{R_z} \{ \int_{R_u} \frac{(u_1 - r_1)^{k_l}}{k_l!} \frac{(u_2 - r_2)^{k_l}}{k_l!} H_{\alpha^*-}(k_0, \ldots, k_{l-1}, r) dW_r \} du$$

$$= \int_{R_z} H_{\alpha^*-}(k_0, \ldots, k_{l-1}, r) (\int_{R_z} 1_{[0,u]}(r) \frac{(u_1 - r_1)^{k_l}}{k_l!} \frac{(u_2 - r_2)^{k_l}}{k_l!} du) dW_r$$

$$= \int_{R_z} H_{\alpha^*-}(k_0, \ldots, k_{l-1}, r) \frac{(s - r_1)^{k_l+1}}{(k_l + 1)!} \frac{(t - r_2)^{k_l+1}}{(k_l + 1)!} dW_r$$

$$= H_{\alpha^*}(k_0, \ldots, k_{l-1}, k_l + 1, z).$$

In the sequel $k_0(\alpha)$ denotes the number of the first components of $\alpha \in \mathcal{M}$ which are equal to 0 until the first 1 or until the end of α if there are only zeros. Furthermore, $k_i(\alpha)$, $i \in \{1, \ldots, l(\alpha^*)\}$, counts the number of components which are equal to 0 between the i-th 1 and the $(i+1)$-th 1 or end of $\alpha \in \mathcal{M}$.

Lemma 6.4.2. *For all* $\alpha \in \mathcal{M}$, *and* $z \in T$, *we have*

$$I_\alpha(z) = H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)}(\alpha), z), \quad P\text{-a.s.}$$

PROOF: We prove the result by induction with respect to $l(\alpha)$.

I. $l(\alpha) = 0$.

Note $\alpha = \alpha^* = v$. By (6.4.1) and (6.4.2)

$$I_\alpha(z) = H_{\alpha^*}(k_0(\alpha), z) = 1.$$

II. $l(\alpha) = l \geq 1$, $\alpha = (j_1, j_2, \ldots, j_l)$.

(*a*) $j_l = 1$.

We have $\alpha\text{-}^* = \alpha^*\text{-}$, $k_i(\alpha\text{-}) = k_i(\alpha)$, $i = 0, 1, \ldots, l(\alpha^*) - 1$, and $k_{l(\alpha^*)} = 0$. By the induction assumption, (6.4.1) and (6.4.2) we have

$$I_\alpha(z) = \int_{R_z} I_{\alpha\text{-}}(u)dW_u = \int_{R_z} H_{\alpha^*\text{-}}(k_0(\alpha), \ldots, k_{l(\alpha^*)\text{-}1}(\alpha), u)dW_u$$

$$= H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)\text{-}1}(\alpha), 0, z).$$

(b) $j_l = 0$.

Note $k_{l(\alpha^*)} \geq 1$ and $\alpha^* = \alpha\text{-}^*$. By the induction assumption and Lemma 6.4.1 we have

$$I_\alpha(z) = \int_{R_z} I_{\alpha\text{-}}(u)du$$

$$= \int_{R_z} H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)}(\alpha) - 1, u)du$$

$$= H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)}(\alpha), z).$$

Lemma 6.4.3. *For $\alpha, \beta \in \mathcal{M}$ with $\alpha^* = \beta^*$, we have*

$$E[I_\alpha(z)I_\beta(z)] = \Big(\prod_{i=0}^{l(\alpha^*)} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)}\Big)^2 \frac{(ts)^{\sigma(\alpha,\, \beta)}}{(\sigma(\alpha, \beta)!)^2} \qquad (6.4.3)$$

where

$$\sigma(\alpha, \beta) = \sum_{i=0}^{l(\alpha^*)} (k_i(\alpha) + k_i(\beta)) + l(\alpha^*).$$

PROOF: We prove the proposition by induction with respect to $l(\alpha^*)$.

1. $l(\alpha^*) = l(\beta^*) = 0$.

Obviously, we have $\alpha^* = \beta^* = v$. Let $l(\alpha) = k \geq 1$, $l(\beta) = k' \geq 1$, then

$$E[I_\alpha(z)I_\beta(z)]$$

$$= \Big(\int_{\{0<z_1< \dots < z_{k-1}< z_k<z\}} dz_1 \cdots dz_k \Big)\Big(\int_{\{0<z_1< \dots < z_{k'-1}< z_{k'}<z\}} dz_1 \cdots dz_{k'} \Big)$$

$$= \frac{(st)^{k+k'}}{(k!k'!)^2} = \Big(\prod_{i=0}^{l(\alpha^*)} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)} \Big)^2 \frac{(ts)^{\sigma(\alpha,\, \beta)}}{(\sigma(\alpha,\, \beta)!)^2} \, .$$

Analogously we get the result for the case $l(\alpha) = 0$ or $l(\beta) = 0$.

2. $l(\alpha^*) = l(\beta^*) = l \geq 1$.

From Lemma 6.4.2 and (6.4.2), we have

$$I_\alpha(z) = H_{\alpha^*}(k_0(\alpha), \dots , k_l(\alpha), z)$$

$$= \int_{R_z} \frac{(s - u_1)^{k_l(\alpha)}}{k_l(\alpha)!} \frac{(t - u_2)^{k_l(\alpha)}}{k_l(\alpha)!} H_{\alpha^{*}-}(k_0(\alpha), \dots , k_{l-1}(\alpha), u)dW_u$$

so

$$E[I_\alpha(z)I_\beta(z)]$$

$$= E\Big[\int_{R_z} \frac{(s - u_1)^{k_l(\alpha)}}{k_l(\alpha)!} \frac{(t - u_2)^{k_l(\alpha)}}{k_l(\alpha)!} H_{\alpha^{*}-}(k_0(\alpha), \dots , k_{l-1}(\alpha), u)dW_u$$

$$\times \int_{R_z} \frac{(s - u_1)^{k_l(\beta)}}{k_l(\beta)!} \frac{(t - u_2)^{k_l(\beta)}}{k_l(\beta)!} H_{\beta^{*}-}(k_0(\beta), \dots , k_{l-1}(\beta), u)dW_u\Big]$$

$$= \int_{R_z} \frac{(s - u_1)^{k_l(\alpha)+k_l(\beta)}(t - u_2)^{k_l(\alpha)+k_l(\beta)}}{(k_l(\alpha)!k_l(\beta)!)^2}$$

$$\times E[H_{\alpha^{*}-}(k_0(\alpha), \dots , k_{l-1}(\alpha), u)H_{\beta^{*}-}(k_0(\beta), \dots , k_{l-1}(\beta), u)]du$$

$$= \int_{R_z} \frac{(s - u_1)^{k_l(\alpha)+k_l(\beta)}(t - u_2)^{k_l(\alpha)+k_l(\beta)}}{(k_l(\alpha)!k_l(\beta)!)^2} \Big(\prod_{i=0}^{l-1} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)} \Big)^2 \frac{(u_1u_2)^{\hat{\sigma}(\alpha,\, \beta)}}{(\hat{\sigma}(\alpha,\, \beta)!)^2} \, du$$

$$= (\prod_{i=0}^{l-1} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)})^2 \frac{1}{(\hat{\sigma}(\alpha, \beta)!)^2} \frac{1}{(k_l(\alpha)!k_l(\beta)!)^2}$$

$$\times \int_{R_z} (s - u_1)^{k_l(\alpha)+k_l(\beta)} (t - u_2)^{k_l(\alpha)+k_l(\beta)} (u_1 u_2)^{\hat{\sigma}(\alpha, \beta)} du$$

where $\hat{\sigma}(\alpha, \beta) = \sum_{i=0}^{l-1} (k_i(\alpha) + k_i(\beta)) + (l - 1)$.

Note

$$\int_{R_z} (s - u_1)^{k_l(\alpha)+k_l(\beta)} (t - u_2)^{k_l(\alpha)+k_l(\beta)} (u_1 u_2)^{\hat{\sigma}(\alpha, \beta)} du$$

$$= [\frac{(k_l(\alpha)+k_l(\beta))!\,\hat{\sigma}(\alpha, \beta)!}{(k_l(\alpha)+k_l(\beta)+\hat{\sigma}(\alpha, \beta)+1)!}]^2 \times (st)^{k_l(\alpha)+k_l(\beta)+\hat{\sigma}(\alpha, \beta)+1},$$

so that (6.4.3) holds.

REMARK: Above arguments follow essentially the lines of [5] in the one-parameter case.

PROPOSITION 6.4.4. *If* $\alpha = (j_1, j_2, \ldots, j_k)$, $\beta = (m_1, m_2, \ldots, m_{k'}) \in \mathcal{M}$, *then for*

$$Q^{\alpha,\beta} \equiv E[\int_{\{0<z_1< \ldots < z_{k-1}< z_k<z\}} C_{j_1}(z_1) \cdots C_{j_{k-1}}(z_{k-1}) C_{j_k}(z_k) dW^{j_1}(z_1) \cdots dW^{j_{k-1}}(z_{k-1}) dW^{j_k}(z_k)$$

$$\times \int_{\{0<z_1<z_2< \ldots < z_{k'}<z\}} C_{m_1}(z_1) \cdots C_{m_{k'-1}}(z_{k'-1}) C_{m_{k'}}(z_{k'}) dW^{m_1}(z_1) \cdots dW^{m_{k'-1}}(z_{k'-1}) dW^{m_{k'}}(z_{k'})]$$

we have

$$Q^{\alpha,\beta} = 0, \qquad \text{for } \alpha^* \neq \beta^* \qquad (6.4.4)$$

and

$$Q^{\alpha,\beta} \leq |C|^{k+k'} (C_{k+k'-2p}^{k-p})^2 \frac{1}{((k+k'-p)!)^2} \qquad \text{for } l(\alpha^*) = l(\beta^*) = p \tag{6.4.5}$$

where $C_0(z) = A_z$, $C_1(z) = B_z$, *and* $|C| = max_i \, sup_{z\in T}|C_i(z)|$.

PROOF: The assertion (6.4.4) follows from the properties of Itô integral immediately.

By Propositions 1.2.4 and 1.2.5 we have

$$Q^{\alpha,\beta} \leq |C|^{k+k'} E(I_\alpha(z) I_\beta(z)). \tag{6.4.6}$$

Note ([5])

$$\prod_{i=0}^{p} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)} \leq C_{(k_0(\alpha)+k_0(\beta))+\cdots+(k_p(\alpha)+k_p(\beta))}^{k_0(\alpha)+\cdots+k_p(\alpha)}$$

$$= C_{k+k'-2p}^{k-p}. \tag{6.4.7}$$

Thus by (6.4.6), (6.4.3) and (6.4.7) we have

$$Q^{\alpha,\beta} \leq |C|^{k+k'} E(I_\alpha(z) I_\beta(z)) = |C|^{k+k'} \Big(\prod_{i=0}^{l(\alpha^*)} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)} \Big)^2 \frac{(ts)^{\sigma(\alpha,\beta)}}{(\sigma(\alpha,\beta)!)^2}$$

$$\leq |C|^{k+k'} (C_{k+k'-2p}^{k-p})^2 \frac{1}{((k+k'-p)!)^2}.$$

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Chapter 7

THE HIDA CALCULUS APPROACH TO MULTIPARAMETER STOCHASTIC INTEGRATION

7.1. Introduction

In this chapter we extend [2] by Al-Hussaini to the multiparameter case. Through homogeneous chaos, Hida's basic operators ∂, ∂^* of white noise calculus, $\dot{W}_t$ derivative of Wiener process W_t at time t and multiplication by $\dot{W}_t$ are introduced on the classical Wiener space $C([0, 1]^d, \mathbb{R})$.

We show that the generalized integral $\int_{T^k} \partial^*_{s_k} \cdots \partial^*_{s_1} \varphi(\underline{s}) d\underline{s}$ coincides with the Skorohod integral $\delta^k \varphi$ on $L^2(\Omega)$. For $d = 2$, a generalized Itô formula is obtained in white noise language. Furthermore we extend the white noise integral [9] by Kuo and Russek to the two-parameter case and establish a series expansion of the Skorohod-Stratonovich integral.

7.2. Definitions and Some Properties of Operators ∂_t, ∂^*_t and D

Let $(\Omega, \mathcal{F}, P)$ be the standard Wiener space over $T = [0, 1]^d$, i.e., $\Omega = C([0, 1]^d, \mathbb{R})$, P is the Wiener measure, and $\mathcal{F}$ is the completion of the Borel σ-field of Ω with respect to P. For every $F \in L^2(\Omega)$, we have the Wiener-Itô chaos expansion [6]

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{7.1.1}$$

where $I_0(f_0) = EF$, and $I_n(f_n)$ denotes the multiple Itô integral of the non-random n-parameter and symmetric function $f_n \in L^2(T^n)$.

Let $\hat{L}^2(T^n)$ be the space of symmetric functions $f_n \in L^2(T^n)$. Put

$$K_n = \{I_n(f_n): f_n \in \hat{L}^2(T^n)\}.$$

Then $L^2(\Omega) = \sum_{n=0}^{\infty} \oplus K_n$. In the sequel duality will be denoted by < , >, sometimes without subscripts. Context should eliminate any confusion.

By analogy with the Hida theory ([4], [5], [7]), we define

DEFINITION 7.2.1. The S–transform on $L^2(\Omega)$ is defined by

$$(UF)(\xi) = E(F\rho(\xi)),$$

where $F \in L^2(\Omega)$, $\xi \in H^{(d+1)/2}(T)$, the Sobolev space of order $\alpha = (d+1)/2$ over T, and

$$\rho(\xi) = \exp\{\int_T \xi(t)dW_t - \frac{1}{2}\int_T \xi^2(t)dt\}.$$

It can be shown [4] that if $\phi \in K_n$ given by $I_n(f_n)$ with $f_n \in \hat{L}^2(T^n)$, then

$$(U\phi)(\xi) = \int_{T^n} f_n(t_1, \dots, t_n)\xi(t_1)\cdots \xi(t_n)dt_1 \cdots dt_n. \tag{7.2.2}$$

To define generalized random variables, we recall some notations of tensor products of Hilbert spaces ([7]).

Let H be real separable Hilbert space with inner product (,). We define by $\hat{\otimes}^n H$ the n–fold symmetric tensor product of H. For fixed $\xi \in H$, let $\xi^{\otimes p}$ denote $\xi \otimes \overset{p}{\cdots} \otimes \xi$. Then the series $\sum_{p \geq 0} (p!)^{-1/2}\xi^{\otimes p}$ defines an element of $\sum_{p \geq 0} \oplus[\hat{\otimes}^p H] \equiv \mathcal{E}_H$ which we denote by $e^{\hat{\otimes}\xi}$. For $\xi, \eta \in H$, we have

$$(e^{\hat{\otimes}\xi}, e^{\hat{\otimes}\eta})_{\mathcal{E}_H} = e^{(\xi, \eta)} .$$

Write $E^0 = L^2(T)$, $E^1 = H^{(d+1)/2}(T)$. E^{-1} is the dual of E^1, and $\mathcal{E}^i = \mathcal{E}_{E^i}$, $i = 0,1,-1$. Then $E^1 \subset E^0 \subset E^{-1}$, $\mathcal{E}^1 \subset \mathcal{E}^0 \subset \mathcal{E}^{-1}$. Put $K_n^{(n)} = \{I_n(f_n)\colon f_n \in \hat{\otimes}^n E^1\}$ and $K_n^{(-n)}$ is the dual of $K_n^{(n)}$. For $\phi \in K_n^{(n)}$ given by $I_n(f_n)$, where $f_n \in \hat{\otimes}^n E^1$, define

$$\|\phi\|_+ = \sqrt{n!}\,\|f_n\|_{\hat{\otimes}^n E^1}.$$

Similarly for corresponding functional $\psi \in K_n^{(-n)}$ represented by $g_n \in \hat{\otimes}^n E^{-1}$, define

$$\|\phi\|_- = \sqrt{n!}\,\|g_n\|_{\hat{\otimes}^n E^{-1}}.$$

Thus we have the triple

$$\begin{tikzcd}
K_n^{(n)} \arrow[r, phantom, "\subset"] \arrow[d] & K_n \arrow[r, phantom, "\subset"] \arrow[d] & K_n^{(-n)} \arrow[d] \\
\sqrt{n!}\,\hat{\otimes}^n E^1 \arrow[r, phantom, "\subset"] & \sqrt{n!}\,\hat{L}^2(T^n) \arrow[r, phantom, "\subset"] & \sqrt{n!}\,\hat{\otimes}^n E^{-1}.
\end{tikzcd}$$

The inclusions are continuous and the arrows are unitary maps. Define

$$(L^2)^+ = \sum_{n=0}^{\infty} \oplus K_n^{(n)}$$

$$(L^2)^- = \sum_{n=0}^{\infty} \oplus K_n^{(-n)},$$

where $K_0^{(0)}$ stands for the real number system. We have

$$(L^2)^+ \subset L^2(\Omega) \subset (L^2)^-.$$

$(L^2)^-$ is called the space of generalized random variables and $(L^2)^+$ is the space of testing random variables.

Note that for each $\xi \in E^1$, the random variable $\rho(\xi)$ is in $(L^2)^+$. Therefore, we can extend the S-transform U to $(L^2)^-$ by

$$(U\psi)(\xi) = <\psi, \rho(\xi)>,$$

$\xi \in E^1$, $\psi \in K_{n.}^{(-n)}$

In what follows we write $U_\phi(\xi)$ in place of $U\phi(\xi)$. Let

$$U_\phi'(\xi, t) = \frac{d}{d\varepsilon} U_\phi(\xi + \varepsilon\delta_t) \mid_{\varepsilon = 0}$$

whenever the right hand side exists. δ_t is the Dirac measure at t.

DEFINITION 7.2.2. Let $\phi \in L^2(\Omega)$. For $t \in T$, suppose $U_\phi'(\ , t)$ exists such that it is an S-transform. Then the operator $\partial_t \phi$ is defined ([4],[5],[7])by

$$\partial_t \phi = U^{-1} U_\phi'(\cdot, t).$$

The adjoint ∂_t^* of ∂_t is defined by

$$<\partial_t^* \psi, \phi> = <\psi, \partial_t \phi>$$

$\psi \in (L^2)^-$, $\phi \in (L^2)^+$.

EXAMPLE 7.2.3. *Let* $\phi \in K_n^{(n)}$ *given by* $\phi = I_n(f_n)$, *where* $f_n \in \hat{\otimes}^n E^1$. *Then*

$$\partial_t \phi = n I_{n-1}(f_n(t, \cdot)). \tag{7.2.3}$$

PROOF: By (7.2.2), for ξ, $h \in E^1$, we have

$$\frac{U_\phi(\xi + \varepsilon h) - U_\phi(\xi)}{\varepsilon} \to n \int_{T^n} f_n(t_1, \dots, t_{n-1}, t)\xi(t_1) \cdots \xi(t_{n-1}) h(t) dt_1 \cdots dt_{n-1} dt$$

$$= U'_\phi(\xi)(h) = (U'_\phi(\xi, \cdot\,), h)_{L^2(T)}$$

as $\varepsilon \to 0$, where

$$U'_\phi(\xi, t) = n\int_{T^{n-1}} f_n(t_1, \dots, t_{n-1}, t)\xi(t_1)\cdots \xi(t_{n-1})dt_1 \cdots dt_{n-1}.$$

Note that

$$|\, U'_\phi(\xi)(h)\,| \le C_1 \,\|h\|,$$

$$|\, U'_\phi(\xi, t)\,| \le C_2\|f_n(\,\cdot\,, t)\|.$$

It holds that

$$U'_\phi(\xi)(h) = \langle J_t(\xi), h\rangle_{E^1, E^{-1}},$$

where

$$J_t(\xi) = \int_T U'_\phi(\xi, s)\delta_t(s)ds \in E^{-1}.$$

Note that $t \to \delta_t \in E^{-1}$ is continuous. Based on Proposition 3.3 of [12], $\|\delta_t - \delta_s\|_{-1} \le C|\, t - s\,|$. Hence (7.2.3) holds.

By a similar calculation we have

$$(\partial_t^* \psi)(\xi) = U^{-1}\xi(t)U_\psi(\xi). \tag{7.2.4}$$

We now introduce the Malliavin-Sobolev derivative [13]. Let $\{e_i, i \ge 1\}$ be a complete orthonormal system in $H \equiv L^2(T)$. For $\phi = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$, let

$$D_{e_i}\phi = \sum_{n=1}^{\infty} nI_{n-1}(\int_T f_n(\cdot, t)e_i(t)dt),$$

provided the series exists in $L^2(\Omega)$.

DEFINITION 7.2.5. Let $\phi \in L^2(\Omega)$ given by $\phi = \sum_{n=0}^{\infty} I_n(f_n)$. Define

$$D\phi = \sum_{i=1}^{\infty} (D_{e_i}\phi)e_i,$$

provided that

$$\sum_{n=0}^{\infty} nn!\|f_n\|^2 < \infty.$$

D is called the Malliavin-Sobolev derivative.

We remark that $D\phi$ as defined above does not depend on the choice of the complete orthonormal set $\{e_i\}$, and $D\phi$ exists as a square integrable H-valued random variable, since

$$E(\|D\phi\|_H^2) = \sum_{i=1}^{\infty} E(D_{e_i}\phi)^2 = \sum_{n=1}^{\infty} n^2(n-1)!\|f_n\|^2 < \infty.$$

Write $\mathcal{D}^{2,1} = \{F \in L^2(\Omega)\colon E(\|D\phi\|_H^2) < \infty\}$.

THEOREM 7.2.6. $\partial = D$ *on* $\mathcal{D}^{2,1}$.

PROOF: We will only consider $\phi = I_n(f_n) \in K_n^{(n)}$. Now

$$D_{e_i}\phi = nI_{n-1}\left(\int_T f_n(\cdot, t)e_i(t)dt\right),$$

$$U_{D_{e_i}\phi}(\xi) = n\int_{T^{n-1}} \left[\int_T f_n(t_1, \dots, t_{n-1}, t)e_i(t)dt\right]\xi(t_1)\cdots\, \xi(t_{n-1})dt_1 \cdots dt_{n-1}$$

$$= U'_{\phi}(\xi)(e_i).$$

So for $h \in E^1$,

$$U_{(D\phi,\, h)}(\xi) = \sum_{i=1}^{\infty} (h, e_i)U_{D_{e_i}\phi}(\xi) = \sum_{i=1}^{\infty} (h, e_i)U'_{\phi}(\xi)(e_i)$$

$$= \sum_{i=1}^{\infty} (h, e_i)(U'_\phi(\xi), e_i) = U'_\phi(\xi)(e_i),$$

thus $D_t\phi = \partial_t\phi$, a.e. t.

7.3. The Multiple Skorohod Integral

Recall the definition of the multiple Skorohod integral defined in chapter 1, we have the following result:

THEOREM 7.3.1. *Suppose* $\phi \in \mathcal{H}^{(k)}(T^k \times \Omega)$. *Then*

$$\int_{T^k} \overset{*}{\partial}_{s_k} \cdots \overset{*}{\partial}_{s_1} \phi(\underline{s}) d\underline{s} \tag{7.3.1}$$

exists and it is equal to $\delta^k\phi$. *Furthermore,*

$$E \left| \int_{T^k} \overset{*}{\partial}_{s_k} \cdots \overset{*}{\partial}_{s_1} \phi(\underline{s}) d\underline{s} \right|^2$$

$$= k! \, \|\tilde{\phi}\|^2_{L^2(T^k \times \Omega)} + \sum_{i=1}^{k} \binom{k}{i} \frac{k!}{i!} E \int_{T^{k+i}} (\partial_{r_1} \cdots \partial_{r_i} \tilde{\phi}(s_1, \dots, s_k))$$

$$\times (\partial_{s_1} \cdots \partial_{s_i} \tilde{\phi}(r_1, \dots, r_i, s_{i+1}, \dots, s_k)) dr_1 \cdots dr_i \, ds_1 \cdots ds_k. \tag{7.3.2}$$

PROOF: Let $\phi(\underline{s}) = \sum_{n=0}^{\infty} \phi_n(\underline{s}) = \sum_{n=0}^{\infty} I_n(f_n(\underline{s}; \cdot))$, where $f_n(\underline{s}; \cdot) \in \hat{L}^2(T^n)$ and $f_n \in L^2(T^{n+k})$. Note

$$\|\overset{*}{\partial}_{s_k} \cdots \overset{*}{\partial}_{s_1} \phi_n\|^2_- = (n+k)! \, \|\delta_{s_k} \hat{\otimes} \cdots \hat{\otimes} \delta_{s_1} \hat{\otimes} f_n(s_1, \dots s_k; \cdot)\|^2_{\otimes^{n+k} E^{-1}}$$

$$\leq c(n+k)! \|f_n(s_1, \dots s_k; \cdot)\|^2_{L^2(T^n)},$$

where $c = c(k)$ is free of n. So

$$\int_{T^k} \|\overset{*}{\partial}_{s_k} \cdots \overset{*}{\partial}_{s_1} \phi_n(s_1, \dots s_n)\|^2_- \, ds_1 \cdots ds_k$$

$$\leq c(n+k)!\ \|f_n\|^2_{L^2(T^{k+n})}.$$

Therefore the integral (7.3.1) exists.

It follows from (7.2.4) that $U(\partial_t^* \psi)(\xi) = \xi(t)U\psi(\xi)$. Thus

$$U\Big(\int_T \cdots \int_T \partial_{s_k}^* \cdots \partial_{s_1}^* \phi_n(s_1, \ldots s_k)ds_1 \cdots ds_k\Big)(\xi)$$

$$= \int_{T^k} U(\partial_{s_k}^* \cdots \partial_{s_1}^* \phi(s_1, \ldots s_k))(\xi)ds_1 \cdots ds_k$$

$$= \int_{T^k} \xi(s_1) \cdots \xi(s_k)\Big(\int_{T^n} f_n(s_1, \ldots s_k; t_1, \ldots t_n)\xi(t_1) \cdots \xi(t_n)dt_1 \cdots dt_n\Big)ds_1 \cdots ds_k$$

$$= \int_{T^k} \int_{T^n} f_n(s_1, \ldots s_k; t_1, \ldots t_n)\xi(s_1) \cdots \xi(s_k)\xi(t_1) \cdots \xi(t_n)dt_1 \cdots dt_n ds_1 \cdots ds_k.$$

So

$$\int_T \cdots \int_T \partial_{s_k}^* \cdots \partial_{s_1}^* \phi_n(s_1, \ldots s_k)ds_1 \cdots ds_k$$

$$= U^{-1}\int_{T^{k+n}} \tilde{f}_n(t_1, \ldots t_k; t_{k+1}, \ldots t_{k+n})\xi(t_1) \cdots \xi(t_{k+n})dt_1 \cdots dt_{k+n}$$

$$= I_{n+1}(\tilde{f}_n)$$

$$= \delta^k \phi_n.$$

The equation (7.3.2) for ϕ follows from Theorem 7.2.8 and Proposition 2.3 of [10].

Recall the fact that the Skorohod integral extends the Itô integral, we then have

COROLLARY 7.3.2. *If $\phi(t)$, $t \in T$ is a nonanticipating process such that*

$$E\int_T |\phi(t)|^2 dt < \infty,$$

then

$$\int_T \phi(t)dW_t = \int_T \partial_t^* \phi(t)dt,$$

where the integral on the left is a d-parameter Itô integral.

Furthermore, if $d = 2$, by Proposition 4.1 of [10] the multiple integral introduced by Wong and Zakai [14] coincides with the generalized integral (7.3.1):

CROLLARY 7.3.3. *Let* $\psi(\omega, z, z')$ *be a function defined on* $\Omega\times T^2$ *such that*

(i) $\psi(z', z'') = 0$, *unless* z' *and* z' *are unordered,*

(ii) ψ *is* $\mathcal{B}(T^2)\times\mathcal{F}$*–measurable and*

$$E\int_{T^2} \psi^2(z, z')dzdz' < \infty$$

(iii) $\psi(z, z')$ *is* $\mathcal{F}_{z\vee z'}$*–adapted.*

Then

$$\int_T \int_T \partial_{z'}^* \partial_{z''}^* \psi(z', z'')dz'dz'' = \int_{T^2} \psi(z', z'')dW_{z'}dW_{z''},$$

the last integral is Wong and Zakai integral [14].

7.4. White Noise Integral

For $t = (t_1, \dots, t_d) \in T$ and $\varepsilon > 0$, write $(t, t+\varepsilon] = \{(s_1, \dots, s_d) \in T: t_i < s_i \leq t_i+\varepsilon, i = 1, 2, \dots, d\}$. Let

$$F_\varepsilon = \frac{1}{\varepsilon^d} I_1(1_{(t,\, t+\varepsilon]}(\cdot)) = \int_T f_\varepsilon(s)dW_s,$$

where $f_\varepsilon = \dfrac{1}{\varepsilon^d} 1_{(t,\, t+\varepsilon]}(\cdot) \in H \subset E^{-1}$.

Thus

$$U_{F_\varepsilon}(\xi) = \int_T f_\varepsilon(s)\xi(s)ds \to <\delta_t, \xi>,$$

as $\varepsilon \to 0$. According to the topology introduced in Section 7.2, the limit $lim_{\varepsilon \to 0} F_\varepsilon = \dot{W}_t$, say, is in $K_1^{(-1)} \subset (L^2)^-$.

Note that

$$U_{\dot{W}_t}(\xi) = <\delta_t, \xi>.$$

By the relation $(\partial_t^* \psi)(\xi) = U^{-1}\xi(t)U\psi(\xi)$, we have

PROPOSITION 7.4.1. $\partial_t^* 1 = \dot{W}_t$.

The multiplication by $\dot{W}_t$ is defined as follows:

DEFINITION 7.4.2. For $\phi \in (L^2)^+$, define $\dot{W}_t\phi$ by

$$<(\dot{W}_t \phi), F> = <\dot{W}_t, \phi F>$$

for any $F \in S_n \equiv \{I_n(f_n)\colon f_n \in C_b^\infty(T^n) \cap \hat{L}^2(T^n)\}, \quad n = 1, 2, \ldots$.

THEOREM 7.4.3. *Let* $\phi \in K_n^{(n)}$. *Then*

$$\dot{W}_t\phi = \partial_t\phi + \partial_t^* \phi.$$

PROOF: Obviously, $\phi \in \mathrm{Dom}\ \partial_t \cap \mathrm{Dom}\ \partial_t^*$. For any $F \in S_m$, it follows from Proposition 7.4.1 that

$$<(\dot{W}_t\phi), F> = <\dot{W}_t, \phi F> = <\partial_t^* 1, \phi F> = <1, \partial_t(\phi F)>$$

$$= <1, \phi\partial_t F> + <1, F\partial_t\phi> = <\partial_t^*\phi, F> + <\partial_t\phi, F>$$

$$= <\partial_t^*\phi + \partial_t\phi, F>.$$

REMARK. Another proof of Theorem 7.4.3 follows from Itô decomposition formula [6]. See [2] for the details.

Theorem 7.4.3 suggests white noise approach to stochastic integration

$$\int_T \dot{W}_z u_z dz = \int_T (\partial_z + \partial_z^*) u_z dz.$$

In the one parameter case, Kuo and Russek [9] exploited trace maps from right and left to define two kinds of integrals $\int_0^1 \partial_{t+} u_t dt$ and $\int_0^1 \partial_{t-} u_t dt$. For $d = 2$, we proceed as follows.

In the sequel we consider $T = [0, 1]^2$. Let $z = (s, t)$ and $z' = (s', t')$ be two points in T. We write $z \leq z'$ if $s \leq s'$ and $t \leq t'$, $z < z'$ if $s < s'$ and $t < t'$, $z \mathbin{\bar{\wedge}} z'$ if $s < s'$ and $t > t'$.

For $k = 1, 2, 3, 4$, and $n \geq 1$, define

$$D_n^1 = \{(z, z_1, \dots, z_n) \in (0, 1)^2 \times (0, 1)^{2n} : z_1 < z\},$$

$$D_n^2 = \{(z, z_1, \dots, z_n) \in (0, 1)^2 \times (0, 1)^{2n} : z_1 > z\},$$

$$D_n^3 = \{(z, z_1, \dots, z_n) \in (0, 1)^2 \times (0, 1)^{2n} : z \mathbin{\bar{\wedge}} z_1\},$$

$$D_n^4 = \{(z, z_1, \dots, z_n) \in (0, 1)^2 \times (0, 1)^{2n} : z_1 \mathbin{\bar{\wedge}} z\}.$$

Let $H^\alpha(D_n^k)$ be the Sobolev space of order $\alpha \geq 0$ on D_n^k, $k = 1, 2, 3, 4$. If $f \in H^\alpha(D_n^1)$, $\alpha > 1$, then f has a version [1] which is denoted by f again such that

$$z = (s, t) \to f(z; (s-, t-), \cdot) \in L^2((0, 1)^2 \times (0, 1)^{2(n-1)}).$$

Furthermore $\underline{z} \to f(z, \underline{z})$ is continuous with values in $H^{\alpha-1}((0, 1)^2 \times (0, 1)^{2(n-1)})$. Define

$$\partial_{z(1)} I_n(f(z;\cdot)) = nI_{n-1}(f(z;(s-,t-)\cdot)),$$

and

$$\dot{W}_{z(1)}\cdot I_n(f(z;\cdot)) = \lim_{s_1\uparrow s, t_1\uparrow t} \dot{W}_{z_1}\cdot I_n(f(z;\cdot)),$$

using the foregoing continuity and replacing ϕ in Theorem 7.4.3 by $I_n(f(z;\cdot))$. Also let

$$\partial_{z(2)} I_n(f(z;\cdot)) = nI_{n-1}(f(z;(s+,t+)\cdot)),$$

$$\dot{W}_{z(2)}\cdot I_n(f(z;\cdot)) = \lim_{s_1\downarrow s, t_1\downarrow t} \dot{W}_{z_1}\cdot I_n(f(z;\cdot)),$$

$$\partial_{z(3)} I_n(f(z;\cdot)) = nI_{n-1}(f(z;(s+,t-)\cdot)),$$

$$\dot{W}_{z(3)}\cdot I_n(f(z;\cdot)) = \lim_{s_1\downarrow s, t_1\uparrow t} \dot{W}_{z_1}\cdot I_n(f(z;\cdot)),$$

$$\partial_{z(4)} I_n(f(z;\cdot)) = nI_{n-1}(f(z;(s-,t+)\cdot)),$$

$$\dot{W}_{z(4)}\cdot I_n(f(z;\cdot)) = \lim_{s_1\uparrow s, t_1\downarrow t} \dot{W}_{z_1}\cdot I_n(f(z;\cdot)).$$

By Theorem 7.2.6 and the trace theorem ([1], [12])

$$\dot{W}_{z(k)}\cdot u_z = \partial_{z(k)} u_z + \partial_z^* u_z.$$

DEFINITION 7.4.4. Let $\phi \in L^2(T\times\Omega)$. Suppose that $\partial_z^*\phi(z)$ and $\partial_{z(k)}\phi(z)$ exist and are integrable on T. We then define

$$\int_T \phi(z)dW(z(k)) = \int_T (\partial_{z(k)}\phi(z) + \partial_z^*\phi(z))dz \qquad (7.4.1)$$

$k = 1, 2, 3, 4.$

Denote by J_k^α, $k = 1, 2, 3, 4$, the set of stochastic processes $u \in L^2(T\times\Omega)$ such that

$$u_z = \sum_{n=0}^{\infty} I_n(f_n(z; \cdot)), \tag{7.4.2}$$

with $f_n \in H^\alpha(D_n^k)$ and

$$\sum_{n=0}^{\infty} (n+1)! \, \| f_n \|^2_{H^\alpha(D_n^k)} < \infty. \tag{7.4.3}$$

THEOREM 7.4.5. *If* $u \in J_k^\alpha$ *with* $\alpha > 1$, *then the stochastic integral* $\int_T u_z dW(z(k))$ *exists in* $L^2(\Omega)$.

PROOF: The idea for the proof is the same as in the one-parameter case [9] but there is an iterated trace theorem considered. Let $u \in J_1^\alpha$ given by (7.4.2) with $f_n \in H^\alpha(D_n^1)$. It follows from the trace theorem ([1], [12]) that the trace

$$f_n(z; z-, z_1, \dots, z_{n-1}) \in L^2((0, 1)^2\times(0, 1)^{2(n-1)}).$$

Thus $\partial_{z(1)} I_n(f_n(z; \cdot))$ exists. By (7.2.3), $\partial_{z(1)} I_n(f_n(z; \cdot)) = nI_{n-1}(f_n(z; z- , \cdot))$.

Note

$$\begin{aligned}
&\int_T E| \partial_{z(1)} I_n(f_n(z; \cdot))|^2 dz \\
&\quad = n^2 \int_T E| I_{n-1}(f_n(z; z- , \cdot))|^2 dz \\
&\quad = nn! \int_T \|f_n(z; z- , \cdot)\|^2_{L^2(T^{n-1})} \, dz \\
&\quad \le \text{const. } nn! \, \| f_n \|^2_{H^\alpha(D_n^1)}.
\end{aligned}$$

By (7.4.3), the integral $\int_T \partial_{z(1)} u_z dz$ exists.

On the other hand,

$$\sum_{n=0}^{\infty} nn! \, \| f_n \|^2_{L^2(T^{n+1})} + \sum_{n=0}^{\infty} n! \, \| f_n \|^2_{L^2(T^{n+1})}$$

$$\leq \sum_{n=0}^{\infty} (n+1)! \, \| f_n \|^2_{H^{\alpha}(D_n^1)} < \infty.$$

By Theorem 7.3.1, $\int_T \partial_z^* \phi(z) dz$ exists. This finishes the proof of the result for $k = 1$. The proofs for $k = 2, 3, 4$ are similar.

THEOREM 7.4.6. *Suppose* u *is a nonanticipating process. If* $E \int_T u_z^2 dz < \infty$, *then for* $k = 2, 3, 4$,

$$\partial_{z(k)} u_z = 0, \quad a.e. \text{ on } T \times \Omega.$$

Therefore

$$\int_T u_z dW(z(k)) = \int_T u_z dW_z, \quad k = 2, 3, 4.$$

The last integral is an Itô integral.

PROOF: Consider $u_z = I_n(f_n(z; \cdot))$. Since u_z is nonanticipating,

$$u_z = I_n(f_n(z; \cdot) 1_{(0, z]^n}(\cdot)).$$

Note that

$$f_n(z; z_1, \ldots, z_n) \, 1_{(0, z]^n}(z_1, \ldots, z_n) = 0, \text{ if } z_1 \notin [0, z].$$

This completes the proof.

7.5. An Itô Formula

In Chapter 2 we established an Itô formula for Skorohod integrals. Then by Theorems 7.2.8 and 7.3.1 we have

THEOREM 7.5.1. *Let* $F \in C_b^4(\mathbb{R})$ *with* $F(x) = 0$. *Suppose* u *is a stochastic process belonging to* $\mathcal{H}^{8,2}(T\times\Omega)\cap\mathcal{H}^{4,3}(T\times\Omega)$. *Let*

$$X_z = \int_{R_z} \overset{*}{\partial}_\zeta u_\zeta d\zeta. \tag{7.5.1}$$

Then

$$\begin{aligned}
F(X_z) &= \int_{R_z} \overset{*}{\partial}_\zeta(F'(X_\zeta)u_\zeta d\zeta \\
&+ \int_{R_z} F''(X_\zeta)u_\zeta[\tfrac{1}{2}u_\zeta + \int_{R_\zeta} \overset{*}{\partial}_{\zeta'}(\partial_\zeta u_{\zeta'})d\zeta']d\zeta \\
&+ \int_{R_z}\int_{R_z} I(\zeta'\overline{\wedge}\zeta)\overset{*}{\partial}_\zeta\overset{*}{\partial}_{\zeta'}(F''(X_{\zeta\vee\zeta'})u_\zeta u_{\zeta'})d\zeta d\zeta' \\
&+ \int_{R_z}\int_{R_z} 1_{\hat{T}^2}(\zeta,\zeta')\overset{*}{\partial}_\zeta\{F''(X_{\zeta\vee\zeta'})u_\zeta\partial_\zeta u_{\zeta'} \\
&+ F^{(3)}(X_{\zeta\vee\zeta'})u_\zeta u_{\zeta'}[\tfrac{1}{2}u_{\zeta'} + \int_{R_{\zeta\vee\zeta}} \overset{*}{\partial}_\gamma(\partial_{\zeta'}u_\gamma)d\gamma]\}d\zeta d\zeta' \\
&+ \int_{R_z}\int_{R_z} 1_{\hat{T}^2}(\zeta,\zeta')\{\tfrac{1}{2}F''(X_{\zeta\vee\zeta'})\partial_\zeta u_\zeta\partial_\zeta u_{\zeta'} \\
&+ F^{(3)}(X_{\zeta\vee\zeta'})u_\zeta[\partial_\zeta u_{\zeta'}\int_{R_{\zeta\vee\zeta}} \overset{*}{\partial}_\gamma(\partial_{\zeta'}u_\gamma)d\gamma + u_{\zeta'}\partial_{\zeta'}u_\zeta + \tfrac{1}{2}u_{\zeta'}\int_{R_{\zeta\vee\zeta}} \overset{*}{\partial}_\gamma(\partial_{\zeta'}\partial_\zeta u_\gamma)d\gamma] \\
&+ F^{(4)}(X_{\zeta\vee\zeta'})u_\zeta u_{\zeta'}[\tfrac{1}{2}u_{\zeta'}\int_{R_{\zeta\vee\zeta}} \overset{*}{\partial}_\gamma(\partial_{\zeta'}u_\gamma)d\gamma + \tfrac{1}{8}u_\zeta u_\zeta \\
&+ \tfrac{1}{2}\int_{R_{\zeta\vee\zeta}} \overset{*}{\partial}_\gamma(\partial_\zeta u_\gamma)d\gamma\int_{R_{\zeta\vee\zeta}} \overset{*}{\partial}_\gamma(\partial_{\zeta'}u_\gamma)d\gamma]\}d\zeta d\zeta'.
\end{aligned} \tag{7.5.2}$$

REMARK 7.5.2. Similarly, from Theorem 2.3.1 we can obtain the Itô formula for the process X_z defined by the sum of generalized multiple integrals:

$$X_z = X_0 + \int_{R_z} f_\zeta d\zeta + \int_{R_z} \overset{*}{\partial}_\zeta \phi_\zeta d\zeta + \int_{R_z}\int_{R_z} \overset{*}{\partial}_\zeta \overset{*}{\partial}_{\zeta'} \psi(\zeta, \zeta')d\zeta d\zeta'$$

$$+ \int_{R_z}\int_{R_z} \overset{*}{\partial}_{\zeta'} h(\zeta, \zeta')d\zeta d\zeta' + \int_{R_z}\int_{R_z} g(\zeta, \zeta')d\zeta d\zeta'.$$

7.6. The Stratonovich Integral

Let $\sigma = \{z_{kl}\}$ be a partition of T given by $z_{kl} = (s_k, t_l)$, $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_{m'} = 1$. For a stochastic process $\phi(z)$, $z \in T$, let

$$\phi_\sigma = \sum_{k,\, l} \frac{1}{4}(\phi(z_{k+1\, l+1}) + \phi(z_{k\, l+1}) + \phi(z_{k+1\, l}) + \phi(z_{kl}))\Delta_{kl}W,$$

where $\Delta_{kl}W = W(z_{k+1\, l+1}) - W(z_{k\, l+1}) - W(z_{k+1\, l}) + W(z_{kl})$. If the limit of ϕ_σ exists in $L^2(\Omega)$ as $\|\sigma\| = \max_{kl} | z_{k+1\, l+1} - z_{kl} |$ tends to zero, we define the limit denoted by

$$\int_T \phi(z)\circ dW_z\ ,$$

to be the Stratonovich integral of ϕ.

THEOREM 7.6.1. *Let* ϕ *be an* $L^2(\Omega)$*–continuous process. If* $\phi \in J_k^\alpha$ *with* $\alpha > 2$, *then*

$$\sum_{i,\, j} \phi(\overset{k}{z}_{ij})\Delta_{ij}W \to \int_T \phi(z)dW(z(k))$$

in $L^2(\Omega)$, *as* $\|\sigma\| \to 0$, *where* $\overset{1}{z}_{ij} = z_{i+1\, j+1}$, $\overset{2}{z}_{ij} = z_{i\, j}$, $\overset{3}{z}_{ij} = z_{i\, j+1}$, $\overset{4}{z}_{ij} = z_{i+1\, j}$.

PROOF: Consider $k = 1$. Let $\phi(z) = I_n(f_n(z; \cdot)$. Note that $\Delta_{ij}W = I_1(1_{\Delta_{ij}})$, where $\Delta_{ij} = (z_{ij}, z_{i+1\, j+1}]$. Then

$$\phi(z_{i+1\, j+1})\Delta_{ij}W = I_n(f_n(z_{i+1\, j+1}; \cdot))I_1(1_{\Delta_{ij}}).$$

By the Itô decomposition formula [6],

$$\sum_{i,j} \phi(z_{i+1\; j+1})\Delta_{ij}W$$

$$= \sum_{i,j} I_{n+1}(f_n(z_{i+1\; j+1};\, \cdot)\hat{\otimes} 1_{\Delta_{ij}}(\cdot)) \; + \; \sum_{i,j} nI_{n\text{-}1}(\int_{\Delta_{ij}} f_n(z_{i+1\; j+1};\, u\, \cdot\,)du)$$

$$= L_1 \; + \; L_2, \text{ say.}$$

Since $\phi(z)$ is $L^2(\Omega)$–continuous,

$$\sum_{i,j} f_n(z_{i+1\; j+1};\, \cdot)\hat{\otimes} 1_{\Delta_{ij}} \to \tilde{f}_n$$

in $L^2(T^{n+1})$.

Thus

$$L_1 \to I_{n+1}(\tilde{f}_n\,) = \int_T \overset{*}{\partial_z} u_z dz$$

in $L^2(\Omega)$.

Note $\phi \in J_k^{\alpha}$, $\alpha > 2$. By Lemma 3.F [3] we have

$$\sum_{i,j} n \int_{\Delta_{ij}} f_n(z_{i+1\; j+1};\, u\, \cdot\,)du \to n\int_T f_n(z\, ;\, z\text{-}\;\; \cdot\,)dz$$

in $L^2(T^{n\text{-}1})$. From the proof of Theorem 7.4.5,

$$nI_{n\text{-}1}(\int_T f_n(z\, ;\, z\text{-}\;\; \cdot\,)dz$$

$$= n\int_T \partial_{z(1)}\phi(z)dz.$$

Therefore,

$$L_1 \; + \; L_2 \to \int_T \phi(z)dW(z(1)).$$

THEOREM 7.6.2. *Let* ϕ *be an* $L^2(\Omega)$*–continuous process. If* $\phi \in \bigcap_{1 \le k \le 4} J_k^\alpha$ *with* $\alpha > 1$. *Then for any orthonormal system* $\{e_i, i \ge 1\}$ of $L^2((0,1)^2)$, *the following series converges in* $L^2(\Omega)$ *and*

$$\sum_{i=1}^{\infty} a_i(\omega) \int_T e_i(z)dW_z = \frac{1}{4} \sum_{k=1}^{4} \int_T \phi(z)dW(z(k))$$

where

$$a_i(\omega) = \int_T \phi(z, \omega)e_i(z)dz. \tag{7.6.1}$$

REMARK. The proof of this theorem follows the same argument as that of Theorem 6 in [9] for the one-parameter case, therefore omitted.

We have proved:

THEOREM 7.6.3. *Let* ϕ *be an* $L^2(\Omega)$*–continuous process. If* $\phi \in \bigcap_{1 \le k \le 4} J_k^\alpha$ *with* $\alpha > 2$. *Then the Stratonovich integral of* ϕ *exists and*

$$\int_T \phi(z) \circ dW_z = \sum_{i=1}^{\infty} a_i(\omega) \int_T e_i(z)dW_z,$$

where a_i *is given by* (7.6.1).

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Chapter 8

A REPRESENTATION FORMULA FOR POISSON FUNCTIONALS

8.1. Introduction

Let F be a square integrable function of a Poisson process over $T = [0, 1]$. Then F can be represented by a stochastic integral of the form

$$F = EF + \int_0^1 g_s dQ_s, \tag{8.1.1}$$

where $Q_t = N_t - t$ denotes the compensated Poisson process and g is a predictable process of $L^2(T\times\Omega)$.

By the chaos decomposition and the definition of Skorohod integral for Poisson processes over $T = [0, 1]$, equation (8.1.1) can be proved in a simple way with the representation

$$g_s = E[D_sF \mid \mathcal{F}_{s-}], \tag{8.1.2}$$

provided that F belongs to Dom D, where D is the unbounded operator, defined below, over the Poisson space.

The purpose of this chapter is to give an explicit form for the integrand g using the method of the Skorohod integral and apply this result to obtain an integration by parts formula. These expressions are given by Elliott and Tsoi [1], using a perturbation of the rate of a Poisson process.

The Skorohod integral over the Poisson space is introduced in Section 8.2 following the approach of Nualart and Vives [4]. In Section 8.3 we establish an explicit form for the representation formula. In Section 8.4 we apply above results to obtain an integration by parts formula.

8.2. Preliminaries

In this section we review the definition of the Skorohod integral over the Poisson space. The main reference is [4].

Let $(\Omega, \mathcal{F}, P)$ be the standard Poisson space over $T = [0, 1]$. That means $\Omega = \{\omega = \sum_{j=1}^{n} \delta_{t_j},\ n \in \mathbb{N} \cup \{\infty\},\ t_j \in T\}$, $\mathcal{F}_0 = \sigma\{\ p_A(\omega) = \omega(A)\ ,\ A \in \mathcal{B}(T)\}$, P is the probability measure defined over $(\Omega, \mathcal{F}_0)$ such that $N_t(\omega) = P_{[0,t]}(\omega)$, $0 \le t \le 1$, is a standard Poisson process, and $\mathcal{F}$ is the completion of $\mathcal{F}_0$ with respect to P.

For any function $F \in L^2(\Omega, \mathcal{F}, P)$, we have the Poisson-Wiener expansion [2]

$$F = \sum_{m=0}^{\infty} I_m(f_m), \tag{8.2.1}$$

where $I_0(f_0) = EF$, and $I_m(f_m)$ denotes the multiple Itô integral of $f_m \in \hat{L}^2(T^m)$ with respect to Poisson measure:

$$I_m(f_m) = \int_{T_*^m} f(t_1, \dots, t_m) Q(dt_1) \cdots Q(dt_m),$$

where $\hat{L}^2(T^m)$ is the set of symmetric functions in $L^2(T^m)$, $Q_t = N_t - t$ is the compensated Poisson process, and

$$T_*^m = \{(t_1, \dots t_m) \in T^m :\ t_i \neq t_j, \text{ for } \forall\ i \neq j\}.$$

If F has the form (8.2.1), we define the derivative of F, DF as an element of $L^2(T \times \Omega)$ by

$$(DF)_s = \sum_{m=1}^{\infty} m I_{m-1}(f_m(s, \cdot)), \tag{8.2.2}$$

provided that this sum converges in $L^2(T \times \Omega)$. That means DF exists if

$$\sum_{m=1}^{\infty} m m!\ \|f_m\|^2_{L^2(T^{m+1})} < \infty.$$

Suppose $u \in L^2(T \times \Omega)$. For every $s \in T$, we have the representation

$$u_s = \sum_{m=0}^{\infty} I_m(f_m(s,\cdot)).$$

The kernels f_m can be chosen so that they belong to $L^2(T^{m+1})$ and are symmetric functions of $t \in T^m$ for all $s \in T$. If the series

$$\sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m) \tag{8.2.3}$$

converges in $L^2(\Omega)$, where $\tilde{f}_m$ denotes the symmetrization of f_m over its $m+1$ variables, then the sum is denoted by $\delta(u)$ and called the Skorohod integral of u. The domain of δ is the set

$$\text{Dom } \delta = \{u \in L^2(T \times \Omega): \sum_{m=0}^{\infty} (m+1)! \, \| \tilde{f}_m \|^2_{L^2(T^{m+1})} < \infty \}.$$

PROPOSITION 8.2.1 *Let $u \in L^2(T \times \Omega)$ be predictable. Then $u \in$ Dom δ and*

$$\delta(u) = \int_0^1 u \, dQ_s.$$

The relation between the operators D and δ is given by the following equality.

PROPOSITION 8.2.2. *Let $F \in$ Dom D and $u \in$ Dom δ, then*

$$E(F \delta \, u) = E(\int_0^1 u_s D_s F ds) \tag{8.2.4}$$

The next result shows that the derivative operator D coincides with a translation operator Φ.

PROPOSITION 8.2.3. *For every $F \in$ Dom D,*

$$D_t F = \Phi_t F(\omega) \ a.s., \ \forall\, t \in T\text{-}a.e. \tag{8.2.5}$$

where

$$\Phi_t F(\omega) = F(\omega + \delta_t) - F(\omega). \tag{8.2.6}$$

As in [4], let $T_0 = 0, T_1, T_2, \dots$ be the jump times of the Poisson process N. Then

$$\Phi_t T_k = \begin{cases} T_{k-1} - T_k & \text{for } t < T_{k-1} \\ t - T_k & \text{for } T_{k-1} < t < T_k \\ 0 & \text{for } t > T_k \end{cases} \tag{8.2.7}$$

In the sequel we often write

$$\delta(u) = \int_0^1 u_s \delta Q_s$$

$$\delta(u 1_{[0,t]}) = \int_0^t u_s \delta Q_s,$$

provided that the Skorohod integrals exist.

For the propositions, proofs are found in [4].

3. Integral Representation

For any $A \in \mathcal{B}(T)$, let $\mathcal{F}_A$ be the σ-field generated by the family of random variables $\{Q(C),\ C \in \mathcal{B}(T),\ C \subset A\}$, completed with respect to P. For each $t \in T$, we denote by $\mathcal{F}_t = \mathcal{F}_{[0,t]}$, and $\mathcal{F}_{t-} = \mathcal{F}_{[0,t)}$.

PROPOSITION 8.3.1. *Let $F \in Dom\, D$. Then*

$$F = EF + \int_0^1 E[D_s F \mid \mathcal{F}_{s-}] \delta Q_s \tag{8.3.1}$$

PROOF: We will follow the approach of Nualart and Pardoux in the case of Brownian measure [3].

Let $n \geq 1$, $g_n \in \hat{L}^2(T^n)$. By the definition of multiple Itô integral [2] we may write $I_n(g_n)$ as an iterated integral:

$$I_n(g_n) = n! \int_0^1 \Big(\int_0^{t_n} \cdots \int_0^{t_3} \Big(\int_0^{t_2} g_n(t_1, \ldots, t_n) 1_{T_*^n}(t_1, \ldots, t_n) dQ_{t_1} \Big) dQ_{t_2} \Big) \cdots dQ_{t_{n-1}} \Big) dQ_{t_n}.$$

Thus

$$E[I_n(g_n) \mid \mathcal{F}_S]$$

$$= n! \int_0^s \Big(\int_0^{t_n} \cdots \int_0^{t_3} \Big(\int_0^{t_2} g_n(t_1, \ldots, t_n) 1_{T_*^n}(t_1,\ldots,t_n) dQ_{t_1} \Big) dQ_{t_2} \Big) \cdots dQ_{t_{n-1}} \Big) dQ_{t_n}$$

$$= I_n(g_n 1_{[0,\, s]^n}) \tag{8.3.2}$$

Similarly

$$E[I_n(g_n) \mid \mathcal{F}_{S-}] = I_n(g_n 1_{[0,\, s)^n}) \tag{8.3.3}$$

Now let F be of the form (8.2.1):

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

We have

$$D_s F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(s, \cdot)).$$

By (8.3.3)

$$E[D_s F \mid \mathcal{F}_{S-}] = \sum_{n=1}^{\infty} n I_{n-1}(f_n(s, \cdot) 1_{[0,\, s)^{n-1}}) = \sum_{n=1}^{\infty} n I_{n-1}(f_n(s, \cdot\,) h_n(s, \cdot\,)),$$

where

$$h_n(s, t_1, \dots, t_{n-1}) = 1_{[0,s)^{n-1}}(t_1, \dots, t_{n-1}) \times 1_{T_*^{n-1}}(t_1, \dots, t_{n-1}).$$

Note

$$n\tilde{h}_n(t_1, \dots, t_n) = 1_{T_*^n}(t_1, \dots, t_n),$$

thus

$$\int_0^1 E[D_s F \mid \mathcal{F}_{s-}] \delta Q_s$$

$$= \sum_{n=1}^{\infty} I_n(n f_n \tilde{h}_n) = \sum_{n=1}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} I_n(f_n) - I_0(f_0) = F - EF$$

LEMMA 8.3.2. *Let* $G(T_1, \dots, T_n, \dots)$ *be bounded, and its first partial derivatives be all bounded by a constant* $K > 0$. *If* $G(T_1, \dots, T_n, \dots) \in Dom\, D$, *then*

$$E\,[D_s G(T_1, \dots, T_n, \dots) \mid \mathcal{F}_s\,] = -\sum_{i=1}^{\infty} E[\frac{\partial G}{\partial t_i}(T_1, \dots, T_n, \dots) 1_{(s<T_i<1)} \mid \mathcal{F}_s\,], \qquad (8.3.4)$$

a.s., $\forall\, s \in T-$ *a.e.*

PROOF : First we remark that (see [1]) for each $M > 0$, the partial sum

$$\sum_{i=1}^{M} E[1_{(s<T_i<1)}] = \sum_{i=1}^{M} P(\, s < T_i < 1) \leq \sum_{i=1}^{M} P(N_1 \geq i) \leq 4\, e^{-1}.$$

By hypothesis, the right hand side of (8.3.4) exists.

From Proposition 8.2.3

$$D_s G(T_1, \dots, T_n, \dots) = \Phi_s G(T_1, \dots, T_n, \dots)$$

$$= G(T_1(\omega + \delta_s), \dots, T_n(\omega + \delta_s), \dots) - G(T_1(\omega), \dots, T_n(\omega), \dots)$$

$$= G(T_1(\omega) + T_1(\omega + \delta_s) - T_1(\omega), \dots, T_n(\omega) + T_n(\omega + \delta_s) - T_n(\omega), \dots)$$

$$- G(T_1(\omega), \dots, T_n(\omega), \dots)$$

$$= G(T_1 + \Phi_s T_1 , \dots , T_n + \Phi_s T_n, \dots) - G(T_1, \dots , T_n, \dots)$$

a.s., $\forall\ s \in A$, where $A \in \mathcal{B}(T)$ having Lebesgue measure 1.

Now for $s \in A \cap (0, 1)$, we have

$$E\,[D_s G(T_1, \dots , T_n, \dots) \mid \mathcal{F}_s] \;=\; \sum_{j=1}^{\infty} E[D_s G(T_1, \dots , T_n, \dots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= \sum_{j=1}^{\infty} E[(G(T_1 + \Phi_s T_1 , \dots , T_n + \Phi_s T_n, \dots) - G(T_1, \dots , T_n, \dots))1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= \sum_{j=1}^{\infty} E[(G(T_1, \dots , T_{j-1} , s, T_j , \dots) - G(T_1, \dots , T_{j-1}, T_j , T_{j+1} , \dots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

a.s.

Define

$$N_t^{*} = N_{t+s} - N_s , \quad 0 \le t \le 1 - s.$$

Then $\{N_t^{*}, 0 \le t \le 1 - s\}$ is a standard Poisson process over $[0, 1-s]$ and it is independent of $\mathcal{F}_s$. Let $T_1^{*}, T_2^{*}, \dots , T_n^{*}, \dots$ be the jump times of N^*. On $(T_{j-1} < s < T_j)$, we have

$$T_k^{*} = T_{j-1+k} - s. \tag{8.3.5}$$

For each j, we will show that

$$E[G(T_1, \dots , T_{j-1}, T_j, \dots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= E[G(T_1, \dots , T_{j-1}, s, T_j, \dots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$+ \sum_{i=j}^{\infty} E[\frac{\partial G}{\partial t_i}(T_1, T_2, \dots)1_{(T_{j-1}<s<T_j,\, T_i<1)} \mid \mathcal{F}_s] \tag{8.3.6}$$

Since G is bounded with first bounded derivatives, it suffices to consider G of the form

$$G(t_1, \dots t_{j-1}, t_j, t_{j+1}, \dots) = G_1(t_1, \dots t_{j-1}) \times G_2(t_j, t_{j+1}, \dots)$$

where G_1 and G_2 are as before. By (8.3.5)

$$E[G_1(T_1, \dots , T_{j-1})G_2(T_j, \dots T_{j-1+k}, \dots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= E[G_1(T_1, \dots , T_{j-1})G_2(T_1^* + s, \dots, T_k^* + s, \dots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= G_1(T_1, \dots , T_{j-1})1_{(T_{j-1}<s<T_j)}E[G_2(T_1^* + s, T_2^* + s, \dots)].$$

Note

$$(T_1^* < 1 - s) = \bigcup_{1 \le i < \infty}(T_i^* < 1 - s, T_{i+1}^* = 1 - s), \text{ a.s.}$$

we write

$$E[\frac{\partial G_2}{\partial t_1}(T_1^* + s, T_2^* + s, \dots)1_{(T_1^*<1-s)}]$$

$$= \sum_{i=1}^{\infty} E[\frac{\partial G_2}{\partial t_1}(T_1^* + s, T_2^* + s, \dots)1_{(T_i^*<1-s,\ T_{i+1}^*=1-s)}].$$

Now

$$E[\frac{\partial G_2}{\partial t_1}(T_1^* + s, T_2^* + s, \dots)1_{(T_1^*<1-s,\ T_2^*=1-s)}]$$

$$= \int_0^{1-s} \frac{\partial G_2}{\partial t_1}(y_1 + s, 1, 1, \dots)dy_1\, e^{-(1-s)}$$

$$= E[G_2(T_1^* + s, T_2^* + s, \dots)1_{(T_1^*=1-s)}] - E[G_2(s, T_1^* + s, T_2^* + s, \dots)1_{(T_1^*=1-s)}].$$

Also for $i \geq 2$,

$$E[\frac{\partial G_2}{\partial t_1}(T_1^* + s, T_2^* + s, \ldots)1_{(T_i^*<1-s,\ T_{i+1}^*=1-s)}]$$

$$= \int_0^{1-s}\int_0^{1-s-y_1}\cdots\int_0^{1-s-(y_1+\cdots+y_{i-1})} \frac{\partial G_2}{\partial t_1}(y_1 + s,\ y_1 + y_2 + s,\ y_1 + \cdots + y_i + s,\ 1,\ 1,\ \ldots)$$

$$dy_i \cdots dy_1\, e^{-(1-s)}$$

$$= \int_0^{1-s}\int_0^{1-s-y_1}\cdots\int_0^{1-s-(y_1+\cdots+y_{i-1})} \{\frac{\partial G_2}{\partial y_1}(y_1 + s,\ y_1 + y_2 + s,\ y_1 + \cdots + y_i + s,\ 1,\ 1,\ \ldots)$$

$$- \sum_{j=2}^{i} \frac{\partial G_2}{\partial t_j}(y_1 + s,\ y_1 + y_2 + s,\ y_1 + \cdots + y_i + s,\ 1,\ 1,\ \ldots)\}dy_i \cdots dy_1\, e^{-(1-s)}$$

$$= \int_0^{1-s}\frac{\partial}{\partial y_1}\{\int_0^{1-s-y_1}\int_0^{1-s-(y_1+\cdots+y_{i-1})} G_2(y_1 + s,\ y_1 + y_2 + s,\ y_1 + \cdots + y_i + s,\ 1,\ 1,\ \ldots)$$

$$dy_i \cdots dy_2\}dy_1 e^{-(1-s)}$$

$$+ \int_0^{1-s}\int_0^{1-s-y_1}\cdots\int_0^{1-s-(y_1+\cdots+y_{i-2})} G_2(y_1 + s,\ y_1 + y_2 + s,\ y_1 + \cdots + y_{i-1} + s,\ 1,$$

$$1,\ \ldots)dy_{i-1} \cdots dy_1\, e^{-(1-s)}$$

$$- \sum_{j=2}^{i} \int_0^{1-s}\int_0^{1-s-y_1}\cdots\int_0^{1-s-(y_1+\cdots+y_{i-1})} \frac{\partial G_2}{\partial t_j}(y_1 + s,\ y_1 + y_2 + s,\ y_1 + \cdots + y_i + s,\ 1,$$

$$1,\ \ldots)dy_i \cdots dy_1\, e^{-(1-s)}$$

$$= -\ E[G_2(s,\ T_1^* + s,\ T_2^* + s,\ \ldots)1_{(T_{i-1}^*<1-s,\ T_i^*=1-s)}]$$

$$+\ E[G_2(T_1^* + s,\ T_2^* + s,\ \ldots)1_{(T_{i-1}^*<1-s,\ T_i^*=1-s)}]$$

$$-\sum_{j=2}^{i} E[\frac{\partial G_2}{\partial t_j}(T_1^* + s, T_2^* + s, \ldots)1_{(T_i^*<1\text{-}s,\ T_{i+1}^*=1\text{-}s)}]$$

Consequently,

$$E[\frac{\partial G_2}{\partial t_1}(T_1^* + s, T_2^* + s, \ldots)1_{(T_1^*<1\text{-}s)}]$$

$$= E[G_2(T_1^* + s, T_2^* + s, \ldots)1_{(T_1^*=1\text{-}s)}] - E[G_2(s, T_1^* + s, T_2^* + s, \ldots)1_{(T_1^*=1\text{-}s)}]$$

$$-\sum_{i=2}^{\infty} E[G_2(s, T_1^* + s, T_2^* + s, \ldots)1_{(T_{i\text{-}1}^*<1\text{-}s,\ T_i^*=1\text{-}s)}]$$

$$+\sum_{i=2}^{\infty} E[G_2(T_1^* + s, T_2^* + s, \ldots)1_{(T_{i\text{-}1}^*<1\text{-}s,\ T_i^*=1\text{-}s)}]$$

$$-\sum_{i=2}^{\infty}\sum_{j=2}^{i} E[\frac{\partial G_2}{\partial t_j}(T_1^* + s, T_2^* + s, \ldots)1_{(T_i^*<1\text{-}s,\ T_{i+1}^*=1\text{-}s)}]$$

$$= E[G_2(T_1^* + s, T_2^* + s, \ldots) - E[G_2(s, T_1^* + s, T_2^* + s, \ldots)$$

$$-\sum_{i=2}^{\infty} E[\frac{\partial G_2}{\partial t_i}(T_1^* + s, T_2^* + s, \ldots)1_{(T_i^*<1\text{-}s)}].$$

Thus

$$E[G_2(T_1^* + s, T_2^* + s, \ldots)]$$

$$= E[G_2(s, T_1^* + s, T_2^* + s, \ldots)] + \sum_{i=1}^{\infty} E[\frac{\partial G_2}{\partial t_i}(T_1^* + s, T_2^* + s, \ldots)1_{(T_i^*<1\text{-}s}].$$

Hence,

$$E[G_1(T_1, \ldots, T_{j-1})G_2(T_j, T_{j+1}, \ldots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= E[G_1(T_1, \ldots, T_{j-1})G_2(s, T_1^* + s, T_2^* + s, \ldots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$+ \sum_{i=1}^{\infty} E[G_1(T_1, \ldots, T_{j-1})\frac{\partial G_2}{\partial t_i}(T_1^* + s, T_2^* + s, \ldots)1_{(T_i^*<1-s)}1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= E[G(T_1, \ldots, T_{j-1})G_2(s, T_j, \ldots)1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$+ \sum_{i=j}^{\infty} E[\frac{\partial}{\partial t_i}(G_1(T_1, \ldots, T_{j-1})G_2(T_j, T_{j+1}, \ldots))1_{(T_{j-1}<s<T_j,\, T_i<1)} \mid \mathcal{F}_s].$$

By a usual limiting procedure, (8.3.6) holds. Therefore

$$E[D_sG(T_1, \ldots, T_n, \ldots) \mid \mathcal{F}_s]$$

$$= \sum_{j=1}^{\infty} E[(G(T_1, \ldots, T_{j-1}, s, T_j, \ldots) - G(T_1, T_2, \ldots))1_{(T_{j-1}<s<T_j)} \mid \mathcal{F}_s]$$

$$= -\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} E[\frac{\partial G}{\partial t_i}(T_1, T_2, \ldots)1_{(T_{j-1}<s<T_j,\, T_i<1)} \mid \mathcal{F}_s]$$

$$= -\sum_{i=1}^{\infty} E[\frac{\partial G}{\partial t_i}(T_1, T_2, \ldots)1_{(s<T_i<1)} \mid \mathcal{F}_s], \quad \text{a.s.}$$

THEOREM 8.3.3. *Let* $G(T_1, \ldots, T_n, \ldots)$ *be bounded and its first partial derivatives be all bounded by a constant* $K > 0$. *If* $G(T_1, \ldots, T_n, \ldots) \in \mathrm{Dom}\, D$, *then*

$$G(T_1, \ldots, T_n, \ldots) = EG(T_1, \ldots, T_n, \ldots)$$

$$+\int_0^1 E[-\sum_{i=1}^{\infty} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n, \ldots)1_{(s<T_i<1)} \mid \mathcal{F}_{s-}]\delta Q_s,$$

PROOF: Note

$$E[D_sF \mid \mathcal{F}_{s-}] = E[E(D_sF \mid \mathcal{F}_s) \mid \mathcal{F}_{s-}].$$

The assertion follows from Proposition 8.3.1 and (8.3.4).

LEMMA 8.3.4. *If* $F \in \mathrm{Dom}\, D$, *then* $E(F \mid \mathcal{F}_t) \in \mathrm{Dom}\, D$, *and*

$$D_s(E[F \mid \mathcal{F}_t]) = E[D_sF \mid \mathcal{F}_t]1_{[0,t]}(s),$$

a.e. on $T\times\Omega$.

PROOF: The lemma follows from (8.2.2) and (8.3.2) immediately.

Consider the martingale M defined by

$$M_t = E[G(T_1, \ldots, T_n, \ldots) \mid \mathcal{F}_t].$$

By Lemma 8.3.4 and Theorem 8.3.3, we have

COROLLARY 8.3.5. *Under the condition of Theorem* 3.3, *we have*

$$M_t = EG(T_1, \ldots, T_n, \ldots) + \int_0^t E[-\sum_{i=1}^{\infty} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n, \ldots)1_{(s<T_i<1)} \mid \mathcal{F}_{s-}]\delta Q_s$$

8.4. Integration by Parts

THEOREM8.4.1. *Let the condition of Theorem* 8.3.3 *in force. Then for any adapted process* u *belonging to* $L^2(T\times\Omega)$, *we have*

$$E[\delta u\, G(T_1, \ldots, T_n, \ldots)] = -E[\sum_{i=1}^{\infty} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n, \ldots)1_{(T_i<1)}\int_0^{T_i} u_s ds].$$

PROOF: An argument similar to that of Theorem 7.3 in [3] shows that u belongs to Dom δ. Write $G = G(T_1, \ldots, T_n, \ldots)$. Then from Proposition 8.2.2 and Lemma 8.3.2,

$$E[\delta u G] = E[\int_0^1 u_s D_s G ds]$$

$$= E[\int_0^1 u_s E[D_s G \mid \mathcal{F}_s] ds]$$

$$= - E[\int_0^1 u_s E[\sum_{i=1}^{\infty} \frac{\partial G}{\partial t_i} (T_1, \ldots, T_n, \ldots) 1_{(s<T_i<1)} \mid \mathcal{F}_s] ds$$

$$= - E[\sum_{i=1}^{\infty} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n, \ldots) 1_{(T_i<1)} \int_0^{T_i} u_s ds].$$

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